

10/19/2026

Last time !

Thm. The units in $\mathbb{Z}[\sqrt{2}]$ are

$$\pm \varepsilon_0^n = \pm (1+\sqrt{2})^n, \quad n \in \mathbb{Z}.$$

If $\varepsilon_n = \pm \varepsilon_0^n = x_n + y_n \sqrt{2}$, where $x_n, y_n \in \mathbb{Z}$, then

$$x_n^2 - 2y_n^2 = N(\pm \varepsilon_n) = (N(\varepsilon_0))^n = (-1)^n.$$

So even values of n gives all $x, y \in \mathbb{Z}$ satisfying

$$x^2 - 2y^2 = 1,$$

and odd values of n gives all $x, y \in \mathbb{Z}$ satisfying

$$x^2 - 2y^2 = -1.$$

Units in $\mathbb{Z}[\sqrt{d}]$ and Pell's equations $x^2 - dy^2 = \pm 1$.

Unit $\varepsilon = x + y\sqrt{d}$ $\longrightarrow$ Solution $(x, y) \in \mathbb{Z}^2$ to
in $\mathbb{Z}[\sqrt{d}]$ $\longleftarrow$ $x^2 - dy^2 = \pm 1$.

Suppose $d > 1$ is not a square.

1. $x^2 - dy^2 = 1$ always has a fundamental solution

$\varepsilon_0 = x_0 + y_0\sqrt{d}$, the least unit > 1 , such that every solution

$\varepsilon = x + y\sqrt{d}$ is one of $\pm \varepsilon_0^n$ ($n \in \mathbb{Z}$).

2. $x^2 - dy^2 = -1$ may not have a solution, but when

it does, it has a fundamental solution $\epsilon_0 = x_0 + y_0\sqrt{d}$,
(unit)

the least unit > 1 , such that every solution $\epsilon = x + y\sqrt{d}$

to $x^2 - dy^2 = (-1)^n$ is one of $\pm \epsilon_0^n$ ($n \in \mathbb{Z}$).

Q: How to find the fundamental solution $\epsilon_0 = x_0 + y_0\sqrt{d}$?

Idea 1. Take positive integers y i.e.

$$\pm 1 + dy^2 = \square$$

E.g. Find the fundamental solution to $x^2 - 7y^2 = 1$.

Look at $1 + 7y^2 = x^2$: $y = 1, 2$ X
 $y = 3, x = 8$.

So the fundamental solution is $(8, 3)$.

Hence all the solutions are

$$\pm (8 + 3\sqrt{7})^n, \quad n \in \mathbb{Z}.$$

Idea 2. Use "continued fractions".

Rational numbers: $\frac{7}{4} = 1 + \frac{3}{4} = 1 + \frac{1}{\frac{4}{3}}$

$$= 1 + \frac{1}{1 + \frac{1}{3}}$$

$$= [1; 1, 3]$$

Quadratic irrationals:

$$\sqrt{2} = 1 + \boxed{\sqrt{2} - 1} = 1 + \frac{1}{\frac{1}{\sqrt{2} - 1}} = 1 + \frac{1}{\frac{\sqrt{2} + 1}{(\sqrt{2} - 1)(\sqrt{2} + 1)}}$$

$$= 1 + \frac{1}{\sqrt{2} + 1} = 1 + \frac{1}{2 + \boxed{\sqrt{2} - 1}}$$

$$= 1 + \frac{1}{2 + \frac{1}{\sqrt{2} + 1}} = 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \dots}}}} = [1; \overline{2}]$$

$$\frac{1+\sqrt{5}}{2} = 1 + \frac{\sqrt{5}-1}{2} = 1 + \frac{1}{\frac{2}{\sqrt{5}-1}} = 1 + \frac{1}{\frac{2(\sqrt{5}+1)}{(\sqrt{5}-1)(\sqrt{5}+1)}}$$

Golden Ratio

$$= 1 + \frac{1}{\frac{\sqrt{5}+1}{2}} = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \dots}}}$$

A (simple) continued fraction has the form

$$a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}}} = [a_0; a_1, a_2, a_3, \dots]$$

$$a_0, a_1, a_2, \dots, \geq 0$$

Transcendental numbers !

$$e = 2.71828\dots = 2 + \frac{1}{1 + \frac{1}{2 + \frac{1}{1 + \frac{1}{1 + \frac{1}{4 + \frac{1}{1 + \frac{1}{6 + \frac{1}{1 + \dots}}}}}}}}$$

$$= [2; \underbrace{1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, \dots}]$$

non-periodic

If $d > 1$ is not a square and $x, y > 0$ is a

solution to $x^2 - dy^2 = 1$ (or $x^2 - dy^2 = -1$), then

$$\frac{x}{y} = a_0 + \frac{1}{a_1 + \dots + \frac{1}{a_k}} = [a_0; a_1, \dots, a_k], \text{ which is}$$

a truncation of $\sqrt{d} = [a_0; \underbrace{a_1, \dots, a_k, \dots}]$
periodic
 $= [a_0; \overline{a_1, \dots, a_k}]$

Ex. $x^2 - 13y^2 = 1$

$$\sqrt{13} = \underset{\triangle}{3} + \boxed{(\sqrt{13} - 3)}$$

$$\frac{1}{\sqrt{13}-3} = \frac{\sqrt{13}+3}{(\sqrt{13}-3)(\sqrt{13}+3)} = \frac{\sqrt{13}+3}{4} = \underset{\triangle}{1} + \frac{\sqrt{13}-1}{4}$$

$$\frac{4}{\sqrt{13}-1} = \frac{\sqrt{13}+1}{3} = \underset{\triangle}{1} + \frac{\sqrt{13}-2}{3}$$

$$\frac{3}{\sqrt{13}-2} = \frac{\sqrt{13}+2}{3} = \underset{\triangle}{1} + \frac{\sqrt{13}-1}{3}$$

$$\frac{3}{\sqrt{13}-1} = \frac{\sqrt{13}+1}{4} = \underset{\triangle}{1} + \frac{\sqrt{13}-3}{4}$$

$$\frac{4}{\sqrt{13}-3} = \sqrt{13} + 3 = \underset{\triangle}{6} + \boxed{(\sqrt{13}-3)}$$

So $\sqrt{13} = [3; \overline{1, 1, 1, 1, 6}]$.

Rmk. The continued fraction expansion of $x \in \mathbb{R}$ is

finite iff $x \in \mathbb{Q}$. It is periodic iff x is a

quadratic rational.

Will use this to solve $x^2 - 13y^2 = 1$.