A FAMILY OF ANALOGUES TO THE ROBIN CRITERION

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Abstract. The Robin criterion states that the Riemann hypothesis is equivalent to the inequality $\sigma(n) < e^{\gamma} n \log\log n$ for all $n > 5040$, where $\sigma(n)$ is the sum of divisors of n, and γ is the Euler–Mascheroni constant. Define the family of functions

$$\sigma^{[k]}(n) \coloneqq \sum_{[d_1,\ldots,d_k]=n} d_1 \ldots d_k$$

where $[d_1,\ldots,d_k]$ is the least common multiple of $d_1,\ldots,d_k$. These functions behave asymptotically like $\sigma(n)^k$ as $k \to \infty$. We prove the following analogue of the Robin criterion: for any $k \geq 2$, the Riemann hypothesis holds if and only if $\sigma^{[k]}(n) < \frac{(e^{\gamma} n \log\log n)^k}{\zeta(k)}$ for all $n > 2162160$, where ζ is the Riemann zeta function.

1. Introduction

In 1894, von Sterneck [25] introduced arithmetic functions F of the form

$$F(n) \coloneqq \sum_{[d_1,\ldots,d_k]=n} f_1(d_1)\cdots f_k(d_k),$$

where $[d_1,\ldots,d_k]$ is the least common multiple of $d_1,\ldots,d_k$ and $f_1,\ldots,f_k$ are arithmetic functions. (See also Lehmer [9, 10].) In particular, von Sterneck considered $f_1 = \cdots = f_k = \varphi$, the Euler totient function, in which case F is the Jordan totient function. Note that the definition of F is equivalent to the identity

$$\sum_{n\geq 1} \frac{F(n)}{n^s} = \sum_{d_1,\ldots,d_k\geq 1} \frac{f_1(d_1)\cdots f_k(d_k)}{[d_1,\ldots,d_k]^s}.$$

Taking $f_i(n) = n$ for all $i = 1,\ldots,k$, we make the following definition.

Definition 1.1. For $k \geq 1$ an integer, and $n \in \mathbb{Z}_{>0}$, we define $\sigma^{[1]}(n) \coloneqq n$ and

$$\sigma^{[k]}(n) \coloneqq \sum_{[d_1,\ldots,d_k]=n} d_1 \cdots d_k.$$

The function $\sigma^{[2]}(n)$ is a special case of [4, (5.10)], but the study of the family of functions $\sigma^{[k]}(n)$ appears to be new.

Lehmer [10] proved that for any arithmetic function $f : \mathbb{Z}_{>0} \to \mathbb{R}$, we have

$$\sum_{[d_1,\ldots,d_k]=n} f(d_1)\cdots f(d_k) = \sum_{d\mid n} \mu(n/d) \left(\sum_{\delta\mid d} f(\delta) \right)^k, \tag{1}$$

where μ is the Möbius function. The expression on the right makes sense if the integer k is replaced by any complex number κ. We call this expression the *κth LCM-power of f*. In this paper we will focus on the case real $\kappa > 1$.

Definition 1.2. For κ a positive real number, and $n \in \mathbb{Z}_{>0}$, we define

$$\sigma^{[\kappa]}(n) := \sum_{d \mid n} \mu(n/d)\sigma(d)^{\kappa}, \tag{2}$$

where $\sigma(n)$ is the sum-of-divisors function.

Note that by (1), Definition 1.2 agrees with Definition 1.1 whenever $\kappa = k$ is a positive integer.

The notation $\sigma^{[\kappa]}$ is motivated by the fact that for each $n \in \mathbb{Z}_{>0}$,

$$\sigma^{[\kappa]}(n) \sim \sigma(n)^{\kappa} \text{ as } \kappa \to \infty.$$

Moreover, $\sigma^{[\kappa]}(n)$ approaches $\sigma(n)^{\kappa}$ monotonically from below as $\kappa \to \infty$. Indeed, since $\sigma^{[\kappa]}(n)$ is multiplicative, it suffices to note that

$$\lim_{\kappa \to \infty} \frac{\sigma^{[\kappa]}(p^{\ell})}{\sigma(p^{\ell})^{\kappa}} = \lim_{\kappa \to \infty} \frac{\sigma(p^{\ell})^{\kappa} - \sigma(p^{\ell-1})^{\kappa}}{\sigma(p^{\ell})^{\kappa}} = \lim_{\kappa \to \infty} \left(1 - \left(\frac{\sigma(p^{\ell-1})}{\sigma(p^{\ell})}\right)^{\kappa}\right) = 1.$$

These relations motivate us to examine other properties of $\sigma^{[\kappa]}$ analogous to those of σ^{κ}.

We start by estimating the partial sums of $\sigma^{[\kappa]}(n)$.

Theorem 1.3. *Let $\kappa > 1$. We have*

$$\sum_{n \le x} \sigma^{[\kappa]}(n) = \frac{c(\kappa)}{(\kappa+1)\zeta(\kappa+1)} x^{\kappa+1} + O(x^{\kappa}(\log x)^{\kappa})$$

for all $x \ge 2$, where ζ is the Riemann zeta function,

$$c(\kappa) := \sum_{n \ge 1} \frac{\sigma_{-1}^{[\kappa]}(n)}{n}, \tag{3}$$

and

$$\sigma_{-1}^{[\kappa]}(n) := \sum_{d \mid n} \mu(n/d)\sigma_{-1}(d)^{\kappa} \tag{4}$$

is the κth LCM-power of $1/n$.

We next derive an upper bound for $\sigma^{[\kappa]}(n)$, and thereby prove an analogue of Grönwall's theorem [7, (25)]

$$\limsup_{n \to \infty} \frac{\sigma(n)}{e^{\gamma} n \log\log n} = 1, \tag{5}$$

namely that the maximal order of $\sigma^{[\kappa]}$ is

$$\frac{(e^{\gamma} n \log\log n)^{\kappa}}{\zeta(\kappa)},$$

where γ is the Euler–Mascheroni constant.

Theorem 1.4 (κ-Grönwall's Theorem)**.** *Let $\kappa > 1$ be a real number. We have*

$$\limsup_{n \to \infty} \frac{\zeta(\kappa)\sigma^{[\kappa]}(n)}{(e^{\gamma} n \log\log n)^{\kappa}} = 1.$$

Our next result provides an equivalence between the Riemann hypothesis and an elementary inequality for $\sigma^{[\kappa]}(n)$, analogous to the Robin criterion. The Robin criterion, which was established in 1984 [19], states that the Riemann hypothesis is equivalent to the inequality

$$\sigma(n) < e^{\gamma} n \log\log n \tag{6}$$

for all $n > 5040$. The weaker statement that if the Riemann hypothesis holds then (6) holds for sufficiently large n was already shown by Ramanujan in 1915 [17, 18]. (See [3] for the history.) We prove the following analogue to the Robin criterion.

Theorem 1.5 (κ-Robin criterion)**.** *Let $\kappa > 3/2$ be a real number. The following are equivalent:*

1. *The Riemann hypothesis holds;*
2. *For all n sufficiently large, we have*

$$\sigma^{[\kappa]}(n) < \frac{(e^{\gamma} n \log\log n)^{\kappa}}{\zeta(\kappa)}.$$

If $\kappa \geq 2$, we may replace (2) above with the condition

(2′) *For all $n > 2162160$, we have*

$$\sigma^{[\kappa]}(n) < \frac{(e^{\gamma} n \log\log n)^{\kappa}}{\zeta(\kappa)}. \tag{7}$$

Remark 1.6. We obtain the Robin criterion with the larger threshold $n > 2162160$ in place of $n > 5040$ by multiplying both sides of (7) by $\zeta(\kappa)$, taking the κth roots, and appealing to Lemma 5.3 below. To recover the original version of the Robin criterion, one needs only to verify Robin's inequality (6) numerically (as Robin himself did) for $5040 < n \leq 2162160$.

Let $H_n := \sum_{1 \leq m \leq n} 1/m$ denote the nth harmonic number. It is well-known that

$$H_n = \log n + \gamma + O(1/n)$$

as $n \to \infty$. In 2000, Lagarias provided an alternative formulation of the Robin criterion, establishing the equivalence of the Riemann hypothesis to the inequality

$$\sigma(n) < H_n + e^{\gamma} e^{H_n} \log H_n \tag{8}$$

for all $n > 1$ [8, Theorem 1.1]. We prove the following analogue to Lagarias' criterion.

Theorem 1.7 (κ-Lagarias criterion)**.** *Let $\kappa \geq 4$ be a real number. The following are equivalent:*

1. *The Riemann hypothesis holds;*
2. *For all $n > 1$, we have*

$$\sigma^{[\kappa]}(n) < \frac{\left(H_n + e^{\gamma} e^{H_n} \log H_n\right)^{\kappa}}{\zeta(\kappa)}.$$

Outline and Notation. The remainder of this paper is organized as follows. In Section 2, we recall several classical arithmetic estimates needed for the proofs of our main results. In Section 3, we determine the mean value and extremal orders of the arithmetic function $\sigma^{[\kappa]}(n)$. In Section 4, we define the κ-colossally abundant numbers in analogy with the colossally abundant numbers of Ramanujan [17, 18] and Robin [19], and develop their properties by means of an auxiliary function $F^{[\kappa]}(x, a)$. In Section 5, we leverage Robin's Theorem (Theorem 5.1) to prove (2) $\Rightarrow$ (1) in Theorem 1.5. In Section 6, we use the theory developed in Section 4 to prove (1) $\Rightarrow$ (2) and then (when $\kappa \geq 2$) to prove (1) $\Rightarrow$ (2′) in Theorem 1.5. In Section 7, we establish Theorem 1.7. In Section 8, we give some possible directions for future work. In our appendix, we give a direct proof of our analogue to Robin's theorem in the case $\kappa = 2$.

For any $x \in \mathbb{R}$, we denote the floor of x by $\lfloor x \rfloor$, which is the largest integer not exceeding x, and the ceiling of x by $\lceil x \rceil$, which is the least integer no less than x. The letter p always represents a prime number. We write $\pi(x)$ for the prime counting function and $\theta(x) := \sum_{p \le x} \log p$ for the Chebyshev theta function. We use Landau's big-O notation and Vinogradov's notation $\ll$ interchangeably, and also adopt the standard order notations $o, \gg, \sim$ from analytic number theory. Unless stated otherwise, all implied constants may depend on κ.

2. Arithmetic estimates

In this section, we collect some classical arithmetic estimates which we will require later in the paper.

2.1. Unconditional estimates.

Even without the Riemann hypothesis, we can obtain meaningful bounds for various functions of arithmetic interest.

Lemma 2.1. *For $x \ge 286$, we have*

$$e^{\gamma} \log x \left(1 - \frac{1}{2(\log x)^2}\right) < \prod_{p \le x} \left(1 - p^{-1}\right)^{-1} < e^{\gamma} \log x \left(1 + \frac{1}{2(\log x)^2}\right).$$

Proof. Rosser–Schoenfeld [20, Theorem 8, p. 70]. ☐

Lemma 2.2. *Let $\lambda > 1$ and let $x > 1$. We have*

$$\sum_{p > x} p^{-\lambda} < \frac{1.01624 \lambda x^{1-\lambda}}{(\lambda - 1) \log x}.$$

Proof. We follow the proof in Rosser–Schoenfeld [20, p. 87], using their Theorem 9 without rounding up the value 1.01624 to 1.02. ☐

Lemma 2.3. *Let $\kappa > 1$ be a real number. For $x > 20000$, we have*

$$\frac{1}{\zeta(\kappa)} < \prod_{p \le x} \left(1 - p^{-\kappa}\right) < \frac{1}{\zeta(\kappa)} \exp\left(\frac{1.01624 \kappa x^{1-\kappa}}{\log x} \left(\frac{1}{\kappa - 1} + 0.000052\right)\right).$$

Proof. The left inequality is immediate. By Lemma 2.2, we have

$$\begin{aligned}
\prod_{p \le x} \left(1 - p^{-\kappa}\right) &= \frac{1}{\zeta(\kappa)} \exp\left(\sum_{n \ge 1} \frac{1}{n} \sum_{p > x} p^{-\kappa n}\right) \\
&< \frac{1}{\zeta(\kappa)} \exp\left(\frac{1.01624 \kappa x}{\log x} \sum_{n \ge 1} \frac{x^{-\kappa n}}{\kappa n - 1}\right) \\
&\le \frac{1}{\zeta(\kappa)} \exp\left(\frac{1.01624 \kappa x^{1-\kappa}}{\log x} \left(\frac{1}{\kappa - 1} - \log\left(1 - x^{-\kappa}\right)\right)\right) \\
&< \frac{1}{\zeta(\kappa)} \exp\left(\frac{1.01624 \kappa x^{1-\kappa}}{\log x} \left(\frac{1}{\kappa - 1} + 0.000052\right)\right)
\end{aligned}$$

where the last equality follows because $-\log(1 - x^{-\kappa}) < -\log(1 - 20000^{-1}) < 0.000052$. ☐

Lemma 2.4. *For $x \ge 19421$ we have*

$$|\theta(x) - x| < \frac{x}{8 \log x}.$$

Proof. Schoenfeld [22, Corollary 2*, p. 359]. ☐

Lemma 2.5. *Let* $\kappa \geq 3/2$ *and* $x > 1$*. We have*

$$1 \leq \prod_{p>x} \frac{1-(p+1)^{-\kappa}}{1-p^{-\kappa}} < \exp\left(\frac{1.0779(\kappa+1)}{x^{\kappa}\log x}\right). \tag{9}$$

Proof. The left inequality is immediate. Now recalling Bernoulli's inequality $1+rx \leq (1+x)^r$ for $r \geq 1, x \geq -1$ and that $\log(1+u) < u$ for $-1 < u < 0$, we compute

$$\begin{aligned}\prod_{p>x} \frac{1-(p+1)^{-\kappa}}{1-p^{-\kappa}} &= \exp\left(\sum_{p>x} \log\left(1+\frac{1-\left(1-\frac{1}{p+1}\right)^{\kappa}}{p^{\kappa}-1}\right)\right)\\ &< \exp\left(\sum_{p>x} \frac{1-\left(1-\frac{1}{p+1}\right)^{\kappa}}{p^{\kappa}-1}\right)\\ &< \exp\left(\sum_{p>x} \frac{\kappa}{(p+1)(p^{\kappa}-1)}\right).\end{aligned}$$

We have

$$(p+1)(p^{\kappa}-1) > \frac{2^{3/2}p^{\kappa+1}}{3} > \frac{p^{\kappa+1}}{1.06067},$$

so

$$\prod_{p>x} \frac{1-(p+1)^{-\kappa}}{1-p^{-\kappa}} < \exp\left(\sum_{p>x} \frac{1.06067\kappa}{p^{\kappa+1}}\right).$$

Applying Lemma 2.2, we obtain (9). ☐

2.2. **Conditional estimates.** If the Riemann hypothesis holds, we can obtain stronger bounds in a few cases of interest.

Lemma 2.6. *If the Riemann hypothesis holds, then for* $x \geq 20000$*, we have*

$$\prod_{\sqrt{2x}<p\leq x} \left(1-p^{-2}\right) \leq \exp\left(\frac{-\sqrt{2}}{\sqrt{x}\log x} + \frac{4}{\sqrt{x}(\log x)^2}\right).$$

Proof. Robin [19, Lemma 6]. ☐

Lemma 2.7. *If the Riemann hypothesis holds, then for* $x \geq 599$ *we have*

$$|\theta(x) - x| < \frac{\sqrt{x}(\log x)^2}{8\pi}.$$

As a consequence, for $x > 0$ *we have* $\theta(x) < 1.000081x$*. For* $x \geq 11927$ *we have* $\theta(x) > 0.985x$*.*

Proof. Schoenfeld [22, Theorem 10], Broughan [3, Lemma 3.11], and Rosser–Schoenfeld [21, Corollary to Theorem 6, p. 265]. ☐

Remark 2.8. The bounds $0.985x < \theta(x)$ and $\theta(x) < 1.000081x$ could be refined using [11, Proposition 2.1], but we do not require such improvements for our results.

Lemma 2.9 (Nicolas's bound)**.** *If the Riemann hypothesis holds, then for $x \geq 20000$, we have*

$$\prod_{p \leq x} \left(1 - \frac{1}{p}\right)^{-1} \leq e^{\gamma} \log \theta(x) \exp\left(\frac{2+\beta}{\sqrt{x}\log x} + \frac{\alpha(x)}{\sqrt{x}\,(\log x)^2}\right),$$

where

$$\alpha(x) \coloneqq \frac{(\theta(x)-x)^2\,(1.31+\log x)}{2x^{3/2}} + (\beta - 2) + \frac{8+4\beta}{\log x} + \frac{2\log x}{x^{1/6}} + \frac{(\log 2\pi)\log x}{x^{1/2}}, \tag{10}$$

and $\beta \coloneqq \gamma + 2 - \log 4\pi$.

Proof. Robin [19, Lemma 5]. We note in passing that Broughan inadvertently introduces an extra 2 in front of the term $(\log 2\pi)\log x/x^{1/2}$ when he rederives this result [3, Lemma 7.7]. □

3. ASYMPTOTICS OF $\sigma^{[\kappa]}(n)$

In this section, we provide both the mean value and extremal order for the functions $\sigma^{[\kappa]}(n)$ when $\kappa > 1$.

3.1. Mean value of $\sigma^{[\kappa]}(n)$.

We furnish the mean value for the functions $\sigma^{[\kappa]}(n)$ when $\kappa > 1$, by means of the following proposition.

Proposition 3.1 ([1, Corollary 1, p. 66])**.** *For $\kappa > 0$ we have*

$$\sum_{n \leq x} \sigma^{\kappa}(n) = \frac{c(\kappa)}{\kappa+1} x^{\kappa+1} + x^{\kappa} \sum_{r=0}^{\lceil \kappa/3 \rceil - 1} a_r (\log x)^{\kappa - r} + O\left(x^{\kappa}(\log x)^{2\kappa/3}(\log\log x)^{4\kappa/3}\right)$$

for all $x \geq 3$, where $c(\kappa) = \sum_{n\geq 1} \sigma_{-1}^{[\kappa]}(n)/n$ as in (3), and $a_r = a_r(\kappa)$ are real constants.

We now proceed with the main theorem of this subsection.

Theorem 3.2 (Theorem 1.3)**.** *Let $\kappa > 1$. We have*

$$\sum_{n \leq x} \sigma^{[\kappa]}(n) = \frac{c(\kappa)}{(\kappa+1)\zeta(\kappa+1)} x^{\kappa+1} + x^{\kappa} \sum_{r=0}^{\lceil \kappa/3 \rceil - 1} a_r' (\log x)^{\kappa - r} + O\left(x^{\kappa}(\log x)^{2\kappa/3}(\log\log x)^{4\kappa/3}\right)$$

for all $x \geq 3$, where $a_r' = a_r'(\kappa)$ are real constants.

Proof. We apply Dirichlet's hyperbola method to obtain

$$\sum_{n \leq x} \sigma^{[\kappa]}(n) = \sum_{a \leq y} \mu(a) \sum_{b \leq x/a} \sigma^{\kappa}(b) + \sum_{b \leq x/y} \sigma^{\kappa}(b) \sum_{a \leq x/b} \mu(a) - \sum_{a \leq y} \mu(a) \sum_{b \leq x/y} \sigma^{\kappa}(b),$$

with $y \coloneqq x^{1/\kappa}$.

To estimate the first double sum,

$$\sum_{a \leq x^{1/\kappa}} \mu(a) \sum_{b \leq x/a} \sigma^{\kappa}(b), \tag{11}$$

we use Proposition 3.1 on the inner sum of (11), and find (11) equals

$$\frac{c(\kappa)}{\kappa+1}\sum_{a\le x^{1/\kappa}}\mu(a)\left(\frac{x}{a}\right)^{\kappa+1}+\sum_{r=0}^{\lceil\kappa/3\rceil-1}a_r\sum_{a\le x^{1/\kappa}}\mu(a)\left(\frac{x}{a}\right)^{\kappa}\left(\log\frac{x}{a}\right)^{\kappa-r}+$$
$$O\left(\sum_{a\le x^{1/\kappa}}\left(\frac{x}{a}\right)^{\kappa}\left(\log\frac{x}{a}\right)^{2\kappa/3}\left(\log\log\frac{x}{a}\right)^{4\kappa/3}\right).$$

The final big-O term may be estimated as being

$$O\left(\sum_{a\le x^{1/\kappa}}\left(\frac{x}{a}\right)^{\kappa}(\log x)^{2\kappa/3}(\log\log x)^{4\kappa/3}\right)=O\left(x^{\kappa}(\log x)^{2\kappa/3}(\log\log x)^{4\kappa/3}\sum_{a\le x^{1/\kappa}}\frac{1}{a^{\kappa}}\right)$$
$$=O\left(x^{\kappa}(\log x)^{2\kappa/3}(\log\log x)^{4\kappa/3}\right),$$

since $\kappa>1$. Since

$$\sum_{n\le x}\frac{\mu(n)}{n^{\kappa}}=\frac{1}{\zeta(\kappa)}+O\left(x^{1-\kappa}\right),$$

the main term of (11) is

$$\frac{c(\kappa)}{\kappa+1}x^{\kappa+1}\sum_{a\le x^{1/\kappa}}\frac{\mu(a)}{a^{\kappa+1}}=\frac{c(\kappa)}{(\kappa+1)\zeta(\kappa+1)}x^{\kappa+1}+O\left(x^{\kappa}\right).$$

To estimate each of the terms

$$a_r x^{\kappa}\sum_{a\le x^{1/\kappa}}\frac{\mu(a)}{a^{\kappa}}\left(\log\frac{x}{a}\right)^{\kappa-r},$$

we use Newton's binomial theorem with error term:

$$(\log x-\log a)^{\kappa-r}=\sum_{j=0}^{J}\binom{\kappa-r}{j}(-1)^j(\log a)^j(\log x)^{\kappa-r-j}+O\left((\log a)^{J+1}(\log x)^{\kappa-r-J-1}\right)$$

where for each r we split the series at $J=J_r$ satisfying $\kappa-r-J_r-1\le 2\kappa/3<\kappa-r-J_r$, so that

$$J_r=\lceil\kappa/3\rceil-r-1.$$

Then we have

$$\left(\log\frac{x}{a}\right)^{\kappa-r}=\sum_{j=0}^{J_r}\binom{\kappa-r}{j}(-1)^j(\log a)^j(\log x)^{\kappa-r-j}+O\left((\log a)^{\lceil\kappa/3\rceil-r}(\log x)^{2\kappa/3}\right),$$

yielding the estimate

$$\sum_{a\le x^{1/\kappa}}\frac{\mu(a)}{a^{\kappa}}\left(\log\frac{x}{a}\right)^{\kappa-r}=\sum_{a\le x^{1/\kappa}}\frac{\mu(a)}{a^{\kappa}}\sum_{j=0}^{J_r}\binom{\kappa-r}{j}(-1)^j(\log a)^j(\log x)^{\kappa-r-j}$$
$$+O\left(\sum_{a\le x^{1/\kappa}}\frac{(\log a)^{\lceil\kappa/3\rceil-r}}{a^{\kappa}}(\log x)^{2\kappa/3}\right)$$
$$=\sum_{a\le x^{1/\kappa}}\frac{\mu(a)}{a^{\kappa}}\sum_{j=0}^{J_r}\binom{\kappa-r}{j}(-1)^j(\log a)^j(\log x)^{\kappa-r-j}+O\left((\log x)^{2\kappa/3}\right).$$

Thus we have

$$a_r x^\kappa \sum_{a\le x^{1/\kappa}} \frac{\mu(a)}{a^\kappa}\left(\log\frac{x}{a}\right)^{\kappa-r} =$$

$$a_r x^\kappa \sum_{a\le x^{1/\kappa}} \frac{\mu(a)}{a^\kappa}\sum_{j=0}^{J_r}\binom{\kappa-r}{j}(-1)^j(\log a)^j(\log x)^{\kappa-r-j} + O\left(x^\kappa(\log x)^{2\kappa/3}\right).$$

for $0\le r\le\lceil\kappa/3\rceil-1$. Following the same method we used for our main term, and re-indexing as needed, we see that (11) is already equal to our desired expression,

$$\frac{c(\kappa)}{(\kappa+1)\zeta(\kappa+1)}x^{\kappa+1} + x^\kappa\sum_{r=0}^{\lceil\kappa/3\rceil-1} a'_r(\log x)^{\kappa-r} + O\left(x^\kappa(\log x)^{2\kappa/3}(\log\log x)^{4\kappa/3}\right).$$

Next we show that the two remaining double sums belong to the error term. For the second double sum, we have

$$\sum_{b\le x^{1-1/\kappa}}\sigma^\kappa(b)\sum_{a\le x/b}\mu(a) = \sum_{b\le x^{1-1/\kappa}}\sigma^\kappa(b)\,O\left(\frac{x}{b}\right) = O\left(x\sum_{b\le x^{1-1/\kappa}}\frac{\sigma^\kappa(b)}{b}\right).$$

Now we use partial summation and by Proposition 3.1 the estimate

$$\sum_{n\le x}\sigma^\kappa(n) = O\left(x^{\kappa+1}\right) \tag{12}$$

to obtain

$$\begin{aligned}\sum_{b\le x^{1-1/\kappa}}\frac{\sigma^\kappa(b)}{b} &= \frac{x^{1/\kappa}}{x}\sum_{b\le x/x^{1/\kappa}}\sigma^\kappa(b) + \int_{1^-}^{x/x^{1/\kappa}}\frac{1}{u^2}\sum_{b\le u}\sigma^\kappa(b)\,du\\ &= \frac{x^{1/\kappa}}{x}O\left(\left(\frac{x}{x^{1/\kappa}}\right)^{\kappa+1}\right) + \int_{1^-}^{x/x^{1/\kappa}}\frac{1}{u^2}O\left(u^{\kappa+1}\right)\,du\\ &= O\left(x^{\kappa-1}\right)\end{aligned}$$

Thus we have

$$\sum_{b\le x^{1-1/\kappa}}\sigma^\kappa(b)\sum_{a\le x/b}\mu(a) = O\left(x^\kappa\right).$$

To estimate the final double sum, we use the trivial estimate

$$\sum_{n\le x}\mu(n) = O(x)$$

and the estimate (12) so that

$$\sum_{a\le x^{1/\kappa}}\mu(a)\sum_{b\le x^{1-1/\kappa}}\sigma^\kappa(b) = O(x^{1/\kappa}(x/x^{1/\kappa})^{\kappa+1}) = O\left(x^\kappa\right).$$

This establishes our result. □

3.2. **Mean value of $\sigma^{[2]}(n)$.**

In this subsection, we improve on Theorem 3.2 in the case $\kappa = 2$ by determining explicit constants. To do so, we require another result from [1].

Proposition 3.3 ([1, Theorem 1, Theorem 2, and Lemma 3])**.** *Let $\{a(n)\}_{n=1}^{\infty}$ be a sequence of real numbers satisfying*

$$\sum_{n=1}^{\infty} \frac{a(n)}{n^s} = \zeta(s)\zeta^2(s+1)f(s+1),$$

and suppose moreover that $f(s)$ has a Dirichlet series expansion which is absolutely convergent in the half plane $\sigma > 1-\lambda$ for some $\lambda > 0$. Let

$$\zeta^3(s+1)f(s+1) = \sum_{n=1}^{\infty} \frac{v(n)}{n^s}. \tag{13}$$

Then

$$\sum_{n\le x} a(n) = x\sum_{n\le x}\frac{v(n)}{n} - \frac{1}{2}\sum_{n\le x} v(n) + O((\log x)^{4/3}(\log\log x)^{8/3}) \tag{14}$$

for all $x \ge 3$, where the implicit constant in the error term depends at most on λ.

Remark 3.4. The statement of Proposition 3.3 is significantly simplified from the original: in full generality, we allow an arbitrary positive power α of the ζ function, and the error term on the right-hand side of (14) will also depend on α.

We also require a formulation of the Selberg–Delange method.

Proposition 3.5 ([23, Theorem II.5.2])**.** *Let $\{\, a_n \,\}_{n\ge1}$ be a positive sequence of real numbers, and suppose $F(s) := \sum_{n\ge1} a_n n^{-s}$ is a Dirichlet series and $\alpha \ge 0$ is a real number such that $F(s)\zeta(s)^{-\alpha}$ can be continued as a holomorphic function of $s = \sigma + i\tau$ for $\sigma \ge 1 - c_0/(1+\max(0,\log\tau))$, and $|F(s)\zeta(s)^{-\alpha}| \le M(1+|\tau|)^{1-\delta}$. For $x \ge 3$, we have*

$$\sum_{n\le x} a_n = x(\log x)^{\alpha-1}\left(\sum_{0\le k\le\alpha-1}\frac{\lambda_k(\alpha)}{(\log x)^k} + O\left(Me^{-c_1\sqrt{\log x}}\right)\right)$$

where $c_1 = c_1(\alpha, M, c_0) > 0$, $\lambda_k(\alpha) := \mu_k(\alpha)/\Gamma(\alpha-k)$, and the $\mu_k(\alpha)$ are defined by the Taylor expansion

$$\frac{s^\alpha F(s+1)}{s+1} = \sum_{k\ge0}\mu_k(\alpha)s^k.$$

The implicit constant in the error term depends at most on c_0, α, δ.

Although the results [1, Theorem 1, Theorem 2, and Lemma 3] and [23, Theorem II.5.2] are more general than Proposition 3.3 and Proposition 3.5, these formulations suffice for our purposes.

Lemma 3.6. *We have*

$$\sum_{n\le x}\left(\frac{\sigma(n)}{n}\right)^2 = \frac{5}{2}\zeta(3)x - \frac{1}{4}(\log x)^2 + O\left((\log x)^{4/3}(\log\log x)^{8/3}\right), \tag{15}$$

for all $x \ge 3$.

Proof. We use Ramanujan's identity

$$\sum_{n=1}^{\infty} \frac{\sigma_a(n)\sigma_b(n)}{n^s} = \frac{\zeta(s)\zeta(s-a)\zeta(s-b)\zeta(s-a-b)}{\zeta(2s-a-b)},$$

taking $a = b = 1$, to get

$$\sum_{n=1}^{\infty} \frac{\sigma^2(n)}{n^s} = \frac{\zeta(s)\zeta^2(s-1)\zeta(s-2)}{\zeta(2s-2)}. \tag{16}$$

Recalling that $\sigma_{-1}(n) = \sigma(n)/n$, and by (4) and (16), we have

$$\sum_{n\geq 1} \frac{\sigma_{-1}^{[2]}(n)}{n^s} = \frac{\zeta^2(s+1)\zeta(s+2)}{\zeta(2s+2)}.$$

When $s = 1$, this implies

$$c(2) = \frac{\zeta^2(2)\zeta(3)}{\zeta(4)} = \frac{5}{2}\zeta(3).$$

We apply Proposition 3.3 with $a(n) = \left(\frac{\sigma(n)}{n}\right)^2$ and $\alpha = 2$ to obtain

$$\sum_{n\leq x} \left(\frac{\sigma(n)}{n}\right)^2 \sim \frac{5}{2}\zeta(3)x$$

for x sufficiently large.

We now refine this estimate by applying the Selberg–Delange method to the sequence $a_n = n\sigma_{-1}^{[2]}(n)$ with $\alpha = 2$, $c_0 = 1/2$, arbitrary $\delta \in (0,1)$, and $M = M(\delta)$. Suppose that x is sufficiently large. Here, as before, $\sigma_{-1}^{[2]}(n)$ is defined by (4). We thereby determine asymptotics for

$$\sum_{n\leq x} n\sigma_{-1}^{[2]}(n)$$

using

$$F(s) = \sum_{n=1}^{\infty} \frac{n\sigma_{-1}^{[2]}(n)}{n^s} = \frac{\zeta(s+1)\zeta^2(s)}{\zeta(2s)}.$$

By Proposition 3.5, it suffices to determine the leading constant of the power series

$$\frac{s^2F(s+1)}{s+1} = \frac{s^2\zeta(s+2)\zeta^2(s+1)}{(s+1)\zeta(2s+2)} = \sum_{k\geq 0} \mu_k(2)s^k.$$

Since $s\zeta(s+1) = 1 + O(|s|)$, we have $(s\zeta^2(s+1)) = 1 + O(|s|)$. Thus,

$$\mu_0(2) = \frac{\zeta(2)}{\zeta(2)} = 1, \text{ and}$$

$$\lambda_0(2) = \frac{\mu_0(2)}{\Gamma(2-0)} = 1.$$

We conclude that

$$\sum_{n\leq x} n\sigma_{-1}^{[2]}(n) = x\log x + O(x).$$

By partial summation, we find

$$\sum_{n\leq x} \sigma_{-1}^{[2]}(n) = \frac{1}{2}(\log x)^2 + O(\log x) \tag{17}$$

and

$$\sum_{n\le x}\frac{\sigma_{-1}^{[2]}(n)}{n}=F(2)-\sum_{n>x}\frac{\sigma_{-1}^{[2]}(n)}{n}=\frac{5}{2}\zeta(3)+O\left(\frac{\log x}{x}\right). \tag{18}$$

Substituting (17) and (18) into (14) with $v(n)=\sigma_{-1}^{[2]}(n)$, we obtain

$$\sum_{n\le x}\left(\frac{\sigma(n)}{n}\right)^2=\frac{5}{2}\zeta(3)x-\frac{1}{4}\left(\log x\right)^2+O\left((\log x)^{4/3}(\log\log x)^{8/3}\right),$$

as required. □

Theorem 3.7. *We have*

$$\sum_{n\le x}\sigma^{[2]}(n)=\frac{5}{6}x^3+O\left(x^2(\log x)^{4/3}(\log\log x)^{8/3}\right)$$

for all $x\ge 3$.

Proof. We apply partial summation to (15) to get

$$\sum_{n\le x}\sigma^2(n)=\frac{5}{6}\zeta(3)x^3-\frac{1}{4}x^2\log x+O\left(x^2(\log x)^{4/3}(\log\log x)^{8/3}\right)$$

for all $x\ge 3$. Note that the second-order term $-\frac{1}{4}x^2\log x$ is absorbed into the error term. We then follow the proof of Theorem 3.2 for the special case $\kappa=2$ to note that the leading constant must be divided by $\zeta(3)$ to obtain our result. □

3.3. **An upper bound for $\sigma^{[\kappa]}(n)$.** In this subsection, we provide an upper bound for $\sigma^{[\kappa]}(n)$. Although this bound is not sharp, it is a maximal order, meaning that the limit superior of $\sigma^{[\kappa]}(n)$ divided by the upper bound is 1. As a corollary, we obtain Theorem 1.4, our analogue to Grönwall's theorem. Before proving an upper bound for $\sigma^{[\kappa]}(n)$, we set some notation.

Definition 3.8. For x a nonnegative real number, we define $\mathcal{P}(x)$, the primorial of x, to be the product of the primes less than or equal to x; that is, $\mathcal{P}(x):=\prod_{p\le x}p$. We define the primorial residual $\mathcal{P}^*(x)$ to be the largest prime ℓ such that $\mathcal{P}(\ell)\le x$ if such a prime exists; otherwise, we define $\mathcal{P}^*(x)=0$.

Remark 3.9. The primorial residual takes its name from the following observation: if we restrict the domain of the primorial to the set of primes, then it becomes a residuated mapping, and the primorial residual (with domain restricted to the interval $[2,\infty)$) is its residual [2, page 11]. Under Definition 3.8, however, the primorial is merely quasi-residuated [2, page 9].

Our upper bound for $\sigma^{[\kappa]}(n)$ is loosely inspired by [19, Theorem 2] (as corrected by [3, Theorem 7.13]). Before we prove a global upper bound for $\sigma^{[\kappa]}(n)$, we require a local upper bound.

Lemma 3.10. *Let $\kappa>1$ be a real number. For p prime and $\ell\ge 1$ an integer, we have*

$$\frac{\sigma^{[\kappa]}(p^\ell)}{p^{\kappa\ell}}<\frac{1-p^{-\kappa}}{\left(1-p^{-1}\right)^{\kappa}}.$$

Proof. For $\ell \geq 1$ arbitrary, we compute

$$\frac{\sigma^{[\kappa]}(p^\ell)}{p^{\kappa\ell}} = \frac{\left(1-p^{-\ell-1}\right)^\kappa - \left(p^{-1}-p^{-\ell-1}\right)^\kappa}{\left(1-p^{-1}\right)^\kappa}. \tag{19}$$

The derivative of (19) with respect to ℓ is

$$\frac{\left(1-p^{-\ell-1}\right)^{\kappa-1} - \left(p^{-1}-p^{-\ell-1}\right)^{\kappa-1}}{\left(1-p^{-1}\right)^\kappa}\kappa p^{-\ell-1}\log p,$$

which is positive, so $\frac{\sigma^{[\kappa]}(p^\ell)}{p^{\kappa\ell}}$ is strictly increasing in ℓ. Thus

$$\frac{\sigma^{[\kappa]}(p^\ell)}{p^{\kappa\ell}} < \lim_{\ell\to\infty}\frac{\sigma^{[\kappa]}(p^\ell)}{p^{\kappa\ell}} = \frac{1-p^{-\kappa}}{\left(1-p^{-1}\right)^\kappa}$$

as desired. □

Theorem 3.11. *For any $\kappa > 1$ and $n \geq e^{19183}$, we have*

$$\sigma^{[\kappa]}(n) < \left(e^\gamma n\log\log n + \frac{0.42n}{\log\log n}\right)^\kappa \prod_{p\mid n}\left(1-p^{-\kappa}\right) \tag{20}$$

and

(21)

$$\sigma^{[\kappa]}(n) < \frac{1}{\zeta(\kappa)}\left(e^\gamma n\log\log n + \frac{0.42n}{\log\log n}\right)^\kappa \exp\left(\frac{1.01624\kappa(\log n)^{1-\kappa}}{\log\log n}\left(\frac{1}{\kappa-1}+0.000052\right)\cdot\right.$$
$$\left.\left(1-\frac{0.14}{\log\log n}\right)^{1-\kappa}\left(1+\frac{1}{7(\log\log n)^2}\right)\right).$$

Proof. By Lemma 3.10, we have

$$\sigma^{[\kappa]}(n) < n^k \prod_{p\mid n}\left(1-p^{-\kappa}\right)\left(1-p^{-1}\right)^{-\kappa}. \tag{22}$$

Now suppose $x = \mathcal{P}^*(n)$ so x is prime, and let x_+ be the next prime so that

$$\mathcal{P}(x) \leq n < \mathcal{P}(x_+).$$

We claim that $\omega(n) \leq \pi(x)$, where $\omega(n)$ denotes the number of distinct prime factors of n. For if $\omega(n) \geq \pi(x_+)$, each distinct prime factor of n would be at least as big as each of the primes $\leq x_+$, so $n \geq \mathcal{P}(x_+)$, a contradiction.

To prove (20), we use

$$\prod_{p\mid n}\left(1-p^{-1}\right)^{-1} \leq \prod_{p\leq x}\left(1-p^{-1}\right)^{-1},$$

which is true since each factor is greater than 1, the number of factors on the left is at most the number of factors on the right, and each $p \mid n$ is either $\leq x$, in which case we have matching factors, or for each factor with $p > x$ on the left we can find an unmatched factor on the right that is greater.

From [5, Theorem 5.9] we have

$$\prod_{p\leq x}\left(1-p^{-1}\right)^{-1} < e^\gamma \log x\left(1+\frac{0.10836}{(\log x)^2}\right) \tag{23}$$

for $x \geq 2278382$, and straightforward numerical computation shows that this inequality in fact holds for $x \geq 19421$. We observe that for the prime 19421 we have $\theta(19421) =$

$19182.3\ldots$, so that $n \geq e^{19183} > e^{\theta(19421)}$. Since $x = \mathcal{P}^*(n)$, we have $n \geq \mathcal{P}(x)$ and so $\log n \geq \theta(x)$. Then Lemma 2.4 gives us

$$\log\log n \geq \log\theta(x) \geq \log x + \log\left(1 - \frac{1}{8\log x}\right) > \log x - \frac{1}{7.899\log x},$$

so that our result would follow if we have

$$e^\gamma \log x\left(1 + \frac{0.10836}{(\log x)^2}\right) < e^\gamma\left(\log x - \frac{1}{7.899\log x}\right) + \frac{0.42}{(\log x - 1/(7.899\log x))},$$

which is clearly true by the stronger inequality

$$e^\gamma \log x\left(1 + \frac{0.10836}{(\log x)^2}\right) < e^\gamma\left(\log x - \frac{1}{7.899\log x}\right) + \frac{0.42}{\log x}. \tag{24}$$

This proves (20).

To prove (21), we use the inequality $(1 - p^{-\kappa})(1 - p^{-1})^{-\kappa} > 1$ and argue as before that since the expression on the left decreases with increasing p, we have

$$\prod_{p \mid n}(1 - p^{-\kappa})(1 - p^{-1})^{-\kappa} \leq \prod_{p \leq x}(1 - p^{-\kappa})(1 - p^{-1})^{-\kappa}.$$

Thus,

$$\sigma^{[\kappa]}(n) < n^k \prod_{p \leq x}(1 - p^{-\kappa})(1 - p^{-1})^{-\kappa} < \left(e^\gamma n\log\log n + \frac{0.42n}{\log\log n}\right)^\kappa \prod_{p \leq x}(1 - p^{-\kappa}). \tag{25}$$

By Lemma 2.3, we have

$$\prod_{p \leq x}(1 - p^{-\kappa}) < \frac{1}{\zeta(\kappa)}\exp\left(\frac{1.01624\kappa x^{1-\kappa}}{\log x}\left(\frac{1}{\kappa - 1} + 0.000052\right)\right). \tag{26}$$

Since $n \leq \mathcal{P}(x_+)$, we have $\log n \leq \theta(x) + \log x_+$. We seek an upper bound for $\log x_+$ in terms of x. By Lemma 2.4, we have

$$x_+ - \frac{x_+}{8\log x_+} - \log x_+ \leq \theta(x_+) - \log x_+ = \theta(x) \leq x + \frac{x}{8\log x}.$$

Then

$$\log x_+ + \log\left(1 - \frac{1}{8\log x_+} - \frac{\log x_+}{x_+}\right) < \log x + \log\left(1 + \frac{1}{8\log x}\right).$$

Since $x \mapsto 1/(8\log x) + (\log x)/x$ is decreasing and $x < x_+$, we have

$$\log x_+ + \log\left(1 - \frac{1}{8\log x} - \frac{\log x}{x}\right) < \log x + \log\left(1 + \frac{1}{8\log x}\right).$$

Thus we have the bound

$$\begin{aligned}\log x_+ &< \log x + \log\left(1 + \frac{1}{8\log x}\right) - \log\left(1 - \frac{1}{8\log x} - \frac{\log x}{x}\right)\\ &= \log x + \log\left(1 + \frac{x + 4\log^2 x}{4x\log x - x/2 - 4\log^2 x}\right).\end{aligned}$$

For the remainder of this argument we will resort to numerics where we have monotonicity. For instance, it is easy to show that for $x \geq 19421$ we have

$$\log x_+ < \log x + \frac{0.26}{\log x}.$$

Invoking Lemma 2.4 again, we obtain

$$\log n < x + \frac{x}{8\log x} + \log x + \frac{0.26}{\log x},$$

and thus

$$\log\log n < \log x + \log\left(1 + \frac{1}{8\log x} + \frac{\log x}{x} + \frac{0.26}{x\log x}\right).$$

This time we have for $x \geq 19421$ that

$$\log\log n < \log x + \frac{0.13}{\log x}.$$

Solving for $\log x$, we have

$$\begin{aligned}\log x &> \frac{\log\log n + \sqrt{(\log\log n)^2 - 0.52}}{2}\\ &> \log\log n - \frac{0.14}{\log\log n}\\ &> \log\log n\left(1 + \frac{1}{7(\log\log n)^2}\right)^{-1}\end{aligned}$$

and that

$$x > \exp\left(-\frac{0.14}{\log\log n}\right)\log n > \left(1 - \frac{0.14}{\log\log n}\right)\log n.$$

Hence, the expression inside the exponential in (26) is less than

$$\frac{1.01624\kappa(\log n)^{1-\kappa}}{\log\log n}\left(\frac{1}{\kappa-1} + 0.000052\right)\left(1 - \frac{0.14}{\log\log n}\right)^{1-\kappa}\left(1 + \frac{1}{7(\log\log n)^2}\right).$$

Combining this with (25) and (26) completes the proof of (21). □

If $\kappa \geq 2$, a computation lets us strengthen Theorem 3.11 (see Theorem 6.3 and the remark thereafter).

Corollary 3.12. *Let $\kappa \geq 2$ be a real number. For each integer $n > 2162160$, the inequalities* (20) *and* (21) *hold.*

Theorem 3.11 also yields a family of analogues to Grönwall's Theorem.

Corollary 3.13. *Let $\kappa > 1$ be a real number. We have*

$$\limsup_{n\to\infty} \frac{\zeta(\kappa)\sigma^{[\kappa]}(n)}{(e^{\gamma} n \log\log n)^{\kappa}} = 1.$$

Proof. The inequality (21) furnished by Theorem 3.11 shows that

$$\limsup_{n\to\infty} \frac{\zeta(\kappa)\sigma^{[\kappa]}(n)}{(e^{\gamma} n \log\log n)^{\kappa}} \leq 1,$$

so it suffices to show this bound is asymptotically obtained. We let $a(n) := \mathcal{P}(n)^{t(n)}$, where $(t(n))_{n\geq 1}$ is a sequence of nonnegative integers such that

$$t(n) \to \infty \text{ but } \log t(n) = o(\log n) \text{ as } n \to \infty;$$

for instance, we could take $t(n) = \lfloor \log n \rfloor$. Taking logarithms twice, we see

$$\log\log a(n) = \log\log \mathcal{P}(n) + \log t(n) \sim \log\log \mathcal{P}(n).$$

Now by Lemma 2.1 and Lemma 2.3, we compute

$$\begin{aligned}\sigma^{[\kappa]}(a(n)) &= a(n)^\kappa \prod_{p\le n} \frac{1-p^{-\kappa}}{(1-p^{-1})^\kappa} \cdot \frac{\left(1-p^{-t(n)-1}\right)^\kappa - \left(p^{-1}-p^{-t(n)-1}\right)^\kappa}{1-p^{-\kappa}} \\ &\sim \frac{(e^\gamma a(n)\log\log a(n))^\kappa}{\zeta(\kappa)} \prod_{p\le n} \frac{\left(1-p^{-t(n)-1}\right)^\kappa - \left(p^{-1}-p^{-t(n)-1}\right)^\kappa}{1-p^{-\kappa}},\end{aligned}$$

where the product is over primes p.

For $\kappa > 1$, $p \geq 2$, and $t \geq 1$, the function

$$t \mapsto \frac{(1-p^{-t})^\kappa - (p^{-1}-p^{-t})^\kappa}{1-p^{-\kappa}}$$

is clearly positive and increasing, with a limit of 1, so

$$\prod_{p\le n} \frac{\left(1-p^{-t(n)-1}\right)^\kappa - \left(p^{-1}-p^{-t(n)-1}\right)^\kappa}{1-p^{-\kappa}} \le 1.$$

On the other hand, for n sufficiently large we have

$$\begin{aligned}\prod_{p\le n} \frac{\left(1-p^{-t(n)-1}\right)^\kappa - \left(p^{-1}-p^{-t(n)-1}\right)^\kappa}{1-p^{-\kappa}} &> \prod_{p} \frac{\left(1-p^{-t(n)-1}\right)^\kappa - \left(p^{-1}-p^{-t(n)-1}\right)^\kappa}{1-p^{-\kappa}} \\ &> \prod_{p} \frac{\left(1-p^{-t(n)-1}\right)^\kappa}{1-p^{-\kappa}} \left(1-p^{-\kappa}\left(1+p^{-t(n)}\right)^\kappa\right) \\ &= \frac{\zeta(\kappa)}{\zeta(t(n)+1)} \prod_{p} \left(1-p^{-\kappa}\left(1+p^{-t(n)}\right)^\kappa\right),\end{aligned}$$

and

$$\lim_{n\to\infty} \prod_{p} \left(1-p^{-\kappa}\left(1+p^{-t(n)}\right)^\kappa\right) = \frac{1}{\zeta(\kappa)},$$

so

$$\limsup_{n\to\infty} \frac{\zeta(\kappa)\sigma^{[\kappa]}(a(n))}{(e^\gamma a(n)\log\log a(n))^\kappa} = 1$$

as desired. ☐

Robin proved the following unconditional upper bound for $\sigma(n)$.

Theorem 3.14 ([19, Theorem 2], [3, Theorem 7.13])**.** *For $n \geq 3$, we have*

$$\sigma(n) < e^\gamma n \log\log n + \frac{2n}{3\log\log n}.$$

We show that we can recover Theorem 3.14 from Theorem 3.11.

Proof of Theorem 3.14. Taking κth roots of both sides of (20) and letting $\kappa \to \infty$ proves a stronger version of this inequality for $n > e^{19183}$. The result now follows from a short computation. ☐

We can derive Grönwall's Theorem itself [7, (25)] from Theorem 3.11 as well. Indeed, taking κth roots of both sides of the inequality in Theorem 3.11 and letting $\kappa \to \infty$ gives us

$$\limsup_{n\to\infty} \frac{\sigma(n)}{e^\gamma n \log\log n} \le 1,$$

and the converse inequality may be obtained by considering $\sigma(a_n)$ as in the proof of Corollary 3.13.

Corollary 3.15 (Grönwall's Theorem)**.** *We have*

$$\limsup_{n\to\infty} \frac{\sigma(n)}{e^{\gamma} n \log\log n} = 1.$$

3.4. **A lower bound for $\sigma^{[\kappa]}(n)$.** In this subsection, we provide an elementary but sharp lower bound for $\sigma^{[\kappa]}(n)$. This bound is also a minimal order, meaning that the limit inferior of $\sigma^{[\kappa]}(n)$ divided by the lower bound is 1.

Proposition 3.16. *Let $\kappa > 1$ be a real number. For each integer $n > 1$ we have*

$$\sigma^{[\kappa]}(n) \geq (n+1)^{\kappa} - 1,$$

with equality if and only if n is prime.

Proof. Observe that for any $x > 0$, the function $f_x : [0,\infty) \to [0,\infty)$ defined by $f_x : t \mapsto (x+t)^{\kappa} - t^{\kappa}$ has derivative

$$f_x'(t) = \kappa \left[(x+t)^{\kappa-1} - t^{\kappa-1} \right] > 0$$

so f_x is strictly increasing in $t \geq 0$. Taking $x = p^{\ell}$ and comparing the values of f_x at $t = \sigma(p^{\ell-1})$ and $t = 1$, we have

$$\sigma^{[\kappa]}(p^{\ell}) = \left(p^{\ell} + \sigma(p^{\ell-1})\right)^{\kappa} - \sigma(p^{\ell-1})^{\kappa} \geq \left(p^{\ell}+1\right)^{\kappa} - 1,$$

with equality if and only if $\ell = 1$. This establishes the proposition for $n = p^{\ell}$.

Now we show that

$$\left((m+1)^{\kappa} - 1\right)\left((n+1)^{\kappa} - 1\right) > (mn+1)^{\kappa} - 1$$

for $\kappa > 1$ and $m, n \geq 1$. We again make use of f_x. Taking $x = m+n$ and comparing the values of f_x at $t = mn+1$ and $t = 1$, we have

$$(m+n+mn+1)^{\kappa} - (mn+1)^{\kappa} > (m+n+1)^{\kappa} - 1.$$

Next, taking $x = m$ and comparing the values of f_x at $t = n+1$ and $t = 1$, we obtain

$$(m+n+1)^{\kappa} - (n+1)^{\kappa} > (m+1)^{\kappa} - 1.$$

Adding up these two inequalities and rearranging the terms yield the desired inequality.

Proposition 3.16 now follows by the multiplicativity of $\sigma^{[\kappa]}(n)$ and an easy inductive argument. ☐

4. κ-COLOSSALLY ABUNDANT NUMBERS

In this section, we develop the theory of κ-colossally abundant numbers in analogy with the classical theory of colossally abundant numbers. The following material is inspired by [3, Chapter 6].

Definition 4.1. Let $\epsilon > 0$ be a real number, and define

$$\rho_{\epsilon}^{[\kappa]}(n) := \frac{\sigma^{[\kappa]}(n)}{n^{\kappa(1+\epsilon)}}. \tag{27}$$

We say that a positive integer N is κ-colossally abundant for ϵ if we have

$$\rho_{\epsilon}^{[\kappa]}(N) \geq \rho_{\epsilon}^{[\kappa]}(n) \tag{28}$$

for all positive integers n. If N is κ-colossally abundant for some $\epsilon > 0$, we say that N is κ-colossally abundant.

Under this definition, the usual colossally abundant numbers should be thought of as "∞-colossally abundant", because N is colossally abundant if and only if for some $\epsilon > 0$ we have

$$\frac{\sigma(N)}{N^{1+\epsilon}} = \lim_{\kappa\to\infty} \sqrt[\kappa]{\frac{\sigma^{[\kappa]}(N)}{N^{\kappa(1+\epsilon)}}} \geq \lim_{\kappa\to\infty} \sqrt[\kappa]{\frac{\sigma^{[\kappa]}(n)}{n^{\kappa(1+\epsilon)}}} = \frac{\sigma(n)}{n^{1+\epsilon}}$$

for all positive integers n.

We require the following definition.

Definition 4.2. If $\kappa > 1$ and $\epsilon \in \mathbb{R}$, and $x > 1$ are given, then for $a \geq 1$ we define

$$f_\epsilon^{[\kappa]}(x,a) := x^{-\kappa(1+\epsilon)} \frac{(x^{a+1}-1)^\kappa - (x^a-1)^\kappa}{(x^a-1)^\kappa - (x^{a-1}-1)^\kappa},$$

and $f^{[\kappa]}(x,a) := f_0^{[\kappa]}(x,a)$. We also define $F^{[\kappa]} : (1,\infty)\times[1,\infty) \to (0,\infty)$ to be

$$F^{[\kappa]}(x,a) := \frac{1}{\kappa \log x} \log f^{[\kappa]}(x,a) = \frac{1}{\kappa\log x} \log\left(\frac{(x^{a+1}-1)^\kappa - (x^a-1)^\kappa}{(x^{a+1}-x)^\kappa - (x^a-x)^\kappa}\right).$$

We adopt the convention $F^{[\kappa]}(x,0) := \infty$ for $x > 1$, but do not consider $(x,0)$ to be properly within the domain of $F^{[\kappa]}$.

By construction, we have

$$f_\epsilon^{[\kappa]}(p,a) = \frac{\sigma^{[\kappa]}(p^a)}{p^{\kappa(1+\epsilon)}\sigma^{[\kappa]}(p^{a-1})} = \frac{\rho_\epsilon^{[\kappa]}(p^a)}{\rho_\epsilon^{[\kappa]}(p^{a-1})}, \text{ and} \tag{29}$$

$$F^{[\kappa]}(p,a) = \frac{1}{\kappa\log p}\log\frac{\sigma^{[\kappa]}(p^a)}{p^\kappa\sigma^{[\kappa]}(p^{a-1})} = \frac{1}{\kappa\log p}\log\frac{\rho_\epsilon^{[\kappa]}(p^a)}{\rho_\epsilon^{[\kappa]}(p^{a-1})} - \epsilon \tag{30}$$

for p prime and $a \in \mathbb{Z}_{>0}$.

Equations (29) and (30) suggest that we may extract information about the κ-colossally abundant numbers by understanding $f_\epsilon^{[\kappa]}$ and $F^{[\kappa]}$. We spend the next subsection developing our understanding of $F^{[\kappa]}$.

4.1. The function $F^{[\kappa]}(x,a)$.

In this subsection, we develop our understanding of the function $F^{[\kappa]}(x,a)$, given in Definition 4.2. We demonstrate that $F^{[\kappa]}(x,a)$ is monotonic in its arguments, and deduce some information about its partial inverses.

Theorem 4.3. *The function $F^{[\kappa]}(x,a)$ is continuous and strictly decreasing in x and a, and continuous and strictly increasing in κ. Moreover, for $a \geq 1$ and $x > 1$ respectively, we have*

$$\lim_{x\to\infty} F^{[\kappa]}(x,a) = 0 \text{ and } \lim_{a\to\infty} F^{[\kappa]}(x,a) = 0,$$

as well as

$$\lim_{x\to 1^+} F^{[\kappa]}(x,a) = \infty.$$

We also have

$$\lim_{\kappa\to\infty} F^{[\kappa]}(x,a) = \frac{1}{\log x}\log\left(\frac{x^{a+1}-1}{x^{a+1}-x}\right) =: F(x,a) \tag{31}$$

and

$$\lim_{\kappa\to 1^+} F^{[\kappa]}(x,a) = 0.$$

We remark that $F(x,a)$, defined in (31) plays an important role in the study of colossally abundant numbers [6]. The monotonicity of F in its arguments is essential to that program.

Proposition 4.4. *Let $F(x,a)$ be as in (31). Then F is decreasing in x and a on $(1,\infty)\times(1,\infty)$.*

Proof. Write

$$F(x,a)=\frac{1}{\log x}\log\left(1+\frac{1}{xG(x,a)}\right),$$

where

$$G(x,a)\coloneqq\frac{x^a-1}{x-1}.$$

Note that

$$\frac{\partial G}{\partial x}(x,a)=\frac{ax^{a-1}(x-1)-(x^a-1)}{(x-1)^2}.$$

Applying Lagrange's mean value theorem to the function $t\mapsto t^a-1$, we see that for some $y\in(1,x)$ we have

$$x^a-1=ay^{a-1}(x-1)<ax^{a-1}(x-1),$$

and therefore $\frac{\partial G}{\partial x}>0$ for $x>1$. This shows that $G(x,a)$ is increasing in $x>1$, whence $F(x,a)$ is decreasing in $x>1$.

It is immediate that $G(x,a)$ is increasing in a, whence $F(x,a)$ is decreasing in $a>1$ as well. ☐

Remark 4.5. The monotonicity of $F(x,a)$ was known to Robin [19], at least when a is an integer.

Analogously, our function $F^{[\kappa]}(x,a)$ plays an important role in our study of κ-colossally abundant numbers. So it is not surprising that we need to investigate the monotonicity of $F^{[\kappa]}(x,a)$, but this time in each of the three variables x,a,κ. Despite the simplicity of the statement of Theorem 4.3, its proof is rather involved. We start by proving that for any fixed $a\geq1$ and $\kappa>1$, $F^{[\kappa]}(x,a)$ is strictly decreasing in $x\in(1,\infty)$. It suffices to show

$$f^{[\kappa]}(x,a)=\frac{(x^{a+1}-1)^\kappa-(x^a-1)^\kappa}{(x^{a+1}-x)^\kappa-(x^a-x)^\kappa} \tag{32}$$

is strictly decreasing in $x\in(1,\infty)$.

Proposition 4.6. *Let $a\geq1$ and $\kappa>1$ be positive real numbers. Then $f^{[\kappa]}(x,a)$ is strictly decreasing in $x\in(1,\infty)$.*

Proof. Fix $a\geq1$ and $\kappa>1$, and let $q^{[\kappa]}(x,a)\coloneqq(x^a-1)^{\kappa-1}$ for $x>1$. Then

$$(x^{a+1}-1)^\kappa-(x^a-1)^\kappa=(x^{a+1}-1)q^{[\kappa]}(x,a+1)-(x^a-1)q^{[\kappa]}(x,a), \tag{33}$$

$$(x^{a+1}-x)^\kappa-(x^a-x)^\kappa=x^k\left[(x^a-1)q^{[\kappa]}(x,a)-(x^{a-1}-1)q^{[\kappa]}(x,a-1)\right]. \tag{34}$$

We compute the derivatives of (33) and (34):

$$\frac{\partial}{\partial x}\left[(x^{a+1}-1)^\kappa-(x^a-1)^\kappa\right]=\kappa x^{-1}\left[(a+1)x^{a+1}q^{[\kappa]}(x,a+1)-ax^aq^{[\kappa]}(x,a)\right],$$

$$\frac{\partial}{\partial x}\left[(x^{a+1}-x)^\kappa-(x^a-x)^\kappa\right]=\kappa x^{\kappa-1}\left\{[(a+1)x^a-1]q^{[\kappa]}(x,a)-(ax^{a-1}-1)q^{[\kappa]}(x,a-1)\right\}.$$

Thus we have

$$\begin{aligned}
L^{[\kappa]}(x,a) &:= \frac{\partial}{\partial x}\left[(x^{a+1}-1)^{\kappa}-(x^{a}-1)^{\kappa}\right]\cdot\left[(x^{a+1}-x)^{\kappa}-(x^{a}-x)^{\kappa}\right]\\
&= \kappa x^{\kappa-1}\left[(a+1)x^{a+1}q^{[\kappa]}(x,a+1)-ax^{a}q^{[\kappa]}(x,a)\right]\cdot\big[(x^{a}-1)q^{[\kappa]}(x,a)-\\
&\quad (x^{a-1}-1)q^{[\kappa]}(x,a-1)\big]\\
&= \kappa x^{\kappa-1}\big[(a+1)(x^{2a+1}-x^{a+1})q^{[\kappa]}(x,a+1)q^{[\kappa]}(x,a)-(a+1)(x^{2a}-x^{a+1})\cdot\\
&\quad q^{[\kappa]}(x,a+1)q^{[\kappa]}(x,a-1)-a(x^{2a}-x^{a})q^{[\kappa]}(x,a)^{2}+a(x^{2a-1}-x^{a})\cdot\\
&\quad q^{[\kappa]}(x,a)q^{[\kappa]}(x,a-1)\big]
\end{aligned}$$

and

$$\begin{aligned}
R^{[\kappa]}(x,a) &:= \left[(x^{a+1}-1)^{\kappa}-(x^{a}-1)^{\kappa}\right]\cdot\frac{\partial}{\partial x}\left[(x^{a+1}-x)^{\kappa}-(x^{a}-x)^{\kappa}\right]\\
&= \kappa x^{\kappa-1}\left[(x^{a+1}-1)q^{[\kappa]}(x,a+1)-(x^{a}-1)q^{[\kappa]}(x,a)\right]\cdot\big\{[(a+1)x^{a}-1]q^{[\kappa]}(x,a)\\
&\quad -(ax^{a-1}-1)q^{[\kappa]}(x,a-1)\big\}\\
&= \kappa x^{\kappa-1}\big\{[(a+1)x^{2a+1}-x^{a+1}-(a+1)x^{a}+1]q^{[\kappa]}(x,a+1)q^{[\kappa]}(x,a)\\
&\quad -(ax^{2a}-x^{a+1}-ax^{a-1}+1)q^{[\kappa]}(x,a+1)q^{[\kappa]}(x,a-1)-[(a+1)x^{2a}-\\
&\quad (a+2)x^{a}+1]q^{[\kappa]}(x,a)^{2}+(ax^{2a-1}-x^{a}-ax^{a-1}+1)q^{[\kappa]}(x,a)q^{[\kappa]}(x,a-1)\big\}.
\end{aligned}$$

To show that $f^{[\kappa]}(x,a)$ is strictly decreasing in $x\in(1,\infty)$, it suffices to prove that $R^{[\kappa]}(x,a)-L^{[\kappa]}(x,a)>0$ for all $x>1$, namely,

$$\begin{aligned}
&A(x,a)q^{[\kappa]}(x,a+1)q^{[\kappa]}(x,a)+B(x,a)q^{[\kappa]}(x,a+1)q^{[\kappa]}(x,a-1)+\\
&C(x,a)q^{[\kappa]}(x,a)q^{[\kappa]}(x,a-1)>(x^{a}-1)^{2}q^{[\kappa]}(x,a)^{2}
\end{aligned}\tag{35}$$

for all $x>1$, where

$$\begin{aligned}
A(x,a) &:= ax^{a+1}-(a+1)x^{a}+1,\\
B(x,a) &:= x^{2a}-ax^{a+1}+ax^{a-1}-1,\\
C(x,a) &:= (a-1)x^{a}-ax^{a-1}+1.
\end{aligned}$$

Now we show that $A(x,a),B(x,a),C(x,a)>0$ for all $x,a>1$. For $A(x,a)$ we have $A_x(x,a)=a(a+1)x^{a-1}(x-1)>0$ for all $x>1$, which implies that $A(x,a)$ is strictly increasing in $x\in(1,\infty)$. Thus $A(x,a)>A(1,a)=0$ for all $x>1$. Similarly, we have $C(x,a)>0$ for all $x>1$ whenever $a>1$. For $B(x,a)$ we find

$$B_x(x,a)=2ax^{2a-1}-a(a+1)x^{a}+a(a-1)x^{a-2}=ax^{a-2}[2x^{a+1}-(a+1)x^{2}+(a-1)].$$

Since $a>1$ implies that

$$\frac{\partial}{\partial x}[2x^{a+1}-(a+1)x^{2}+(a-1)]=2(a+1)x(x^{a-1}-1)>0$$

for all $x>1$, so the function $2x^{a+1}-(a+1)x^{2}+(a-1)$ is strictly increasing in $x\in(1,\infty)$. It follows that $B_x(x,a)>0$ for all $x>1$. Hence $B(x,a)$ is strictly increasing in $x\in(1,\infty)$ and $B(x,a)>B(1,a)=0$ for all $x>1$.

Now we prove (35), which can be rewritten as

$$\begin{aligned}
&A(x,a)[q(x,a+1)q(x,a)]^{\kappa-1}+B(x,a)[q(x,a+1)q(x,a-1)]^{\kappa-1}+\\
&\qquad\qquad C(x,a)[q(x,a)q(x,a-1)]^{\kappa-1}>q(x,a)^{2\kappa},
\end{aligned}$$

where $q(x,a) \coloneqq q^{[2]}(x,a) = x^a - 1$. When $a = 1$, the inequality above becomes

$$A(x,1)[q(x,2)q(x,1)]^{\kappa-1} > q(x,1)^{2\kappa},$$

which is true for $x > 1$ due to $A(x,1) = q(x,1)^2$ and $q(x,2) = x^2 - 1 > q(x,1)$. So it remains to consider the case $a > 1$. A straightforward computation shows that

$$\frac{A(x,a)}{q(x,a+1)q(x,a)} + \frac{B(x,a)}{q(x,a+1)q(x,a-1)} + \frac{C(x,a)}{q(x,a)q(x,a-1)} = 1$$

for all $x > 1$. Since $k > 1$ and

$$A(x,a) + B(x,a) + C(x,a) = (x^a - 1)^2 = q(x,a)^2,$$

it follows by the power mean inequality that

$$\begin{aligned} A(x,a)[q(x,a+1)q(x,a)]^{\kappa-1} + B(x,a)[q(x,a+1)q(x,a-1)]^{\kappa-1} + \\ C(x,a)[q(x,a)q(x,a-1)]^{\kappa-1} \geq (A(x,a) + B(x,a) + C(x,a))^{\kappa} = q(x,a)^{2\kappa} \end{aligned}$$

for all $x > 1$ with equality if and only if

$$q(x,a+1)q(x,a) = q(x,a+1)q(x,a-1) = q(x,a)q(x,a-1),$$

or equivalently if and only if $q(x,a+1) = q(x,a) = q(x,a-1)$, which is clearly impossible for $x > 1$. This completes the proof of (35) and hence that of the proposition. □

Next, we prove that $F^{[\kappa]}(x,a)$ is strictly decreasing in $a \in [1,\infty)$ for $x > 1$. Again, it suffices to consider $f^{[\kappa]}(x,a)$.

Proposition 4.7. *Let $x > 1$ and $\kappa > 1$. Then $f^{[\kappa]}(x,a)$ is a strictly decreasing function of $a \in [1,\infty)$.*

Proof. As in the proof of Proposition 4.6, we put $q(x,a) = x^a - 1 > 0$. We compute the partial derivative $\partial f^{[\kappa]}/\partial a$ to obtain

$$x^{\kappa} \cdot \frac{\partial f^{[\kappa]}}{\partial a}(x,a) = \frac{\kappa x^{a-1}\log x}{[q(x,a)^{\kappa} - q(x,a-1)^{\kappa}]^2} v^{[\kappa]}(x,a),$$

where

$$\begin{aligned} v^{[\kappa]}(x,a) &= \left[q(x,a+1)^{\kappa-1}x^2 - q(x,a)^{\kappa-1}x\right]\left[q(x,a)^{\kappa} - q(x,a-1)^{\kappa}\right] \\ &\quad - \left[q(x,a)^{\kappa-1}x - q(x,a-1)^{\kappa-1}\right]\left[q(x,a+1)^{\kappa} - q(x,a)^{\kappa}\right] \\ &= \left[q(x,a+1)^{\kappa-1}x^2 - q(x,a)^{\kappa-1}x\right]\left[q(x,a)^{\kappa-1}(x^a - 1) - q(x,a-1)^{\kappa-1}(x^{a-1} - 1)\right] \\ &\quad - \left[q(x,a)^{\kappa-1}x - q(x,a-1)^{\kappa-1}\right]\left[q(x,a+1)^{\kappa-1}(x^{a+1} - 1) - q(x,a)^{\kappa-1}(x^a - 1)\right] \\ &= x(1-x)[q(x,a+1)q(x,a)]^{\kappa-1} + (x^2 - 1)[q(x,a+1)q(x,a-1)]^{\kappa-1} \\ &\quad + (1-x)[q(x,a)q(x,a-1)]^{\kappa-1}. \end{aligned}$$

Thus it suffices to show $v^{[\kappa]}(x,a) < 0$ for all $a > 1$. Equivalently, we must show that

$$x[q(x,a+1)q(x,a)]^{\kappa-1} + [q(x,a)q(x,a-1)]^{\kappa-1} > (1+x)[q(x,a+1)q(x,a-1)]^{\kappa-1},$$

which can be rewritten as

$$\frac{x}{1+x}\left(\frac{q(x,a)}{q(x,a-1)}\right)^{\kappa-1} + \frac{1}{1+x}\left(\frac{q(x,a)}{q(x,a+1)}\right)^{\kappa-1} > 1. \tag{36}$$

Simple computation shows that

$$\frac{x}{1+x}\left(\frac{q(x,a)}{q(x,a-1)}\right)^{-1} + \frac{1}{1+x}\left(\frac{q(x,a)}{q(x,a+1)}\right)^{-1} = 1.$$

Since $\kappa - 1 > 0$, it follows from the power mean inequality that the left side of (36) is greater than or equal to

$$\left(\frac{x}{1+x}\left(\frac{q(x,a)}{q(x,a-1)}\right)^{-1} + \frac{1}{1+x}\left(\frac{q(x,a)}{q(x,a+1)}\right)^{-1}\right)^{1-\kappa} = 1$$

with equality if and only if

$$\frac{q(x,a)}{q(x,a-1)} = \frac{q(x,a)}{q(x,a+1)}.$$

Since the equality above does not hold for $x > 1$ and $a > 1$, we finish the proof of (36). ☐

For $x, \kappa > 1$ and $a \geq 1$, we have shown that

$$F^{[\kappa]}(x,a) = \frac{\log f^{[\kappa]}(x,a)}{\kappa \log x} = \frac{1}{\kappa \log x} \log \frac{(x^{a+1}-1)^\kappa - (x^a-1)^\kappa}{(x^{a+1}-x)^\kappa - (x^a-x)^\kappa}$$

is strictly decreasing as x or a increases.

The arithmetic mean–geometric mean inequality implies that $x^{a+1} + x^{a-1} > 2x^a$, which is equivalent to

$$(x^{a+1}-1)^\kappa (x^{a-1}-1)^\kappa < (x^a-1)^{2\kappa}. \tag{37}$$

Thus we have

$$\frac{(x^{a+1}-1)^\kappa - (x^a-1)^\kappa}{(x^{a+1}-x)^\kappa - (x^a-x)^\kappa} < \frac{(x^{a+1}-1)^\kappa}{(x^{a+1}-x)^\kappa},$$

which implies that $F^{[\kappa]}(x,a) < F(x,a)$, with F as in (31). In fact, we shall show that as a function of κ, $F^{[\kappa]}(x,a)$ is strictly increasing on $(1,\infty)$. To this end, let us rewrite

$$F^{[\kappa]}(x,a) = \frac{1}{\kappa \log x} \log \frac{q(x,a+1)^\kappa - q(x,a)^\kappa}{q(x,a)^\kappa - q(x,a-1)^\kappa} - 1.$$

We need the following elementary lemma.

Lemma 4.8. *The function*

$$H(t) = \frac{t\,(\log t)^2}{(t-1)^2}$$

is strictly decreasing on $(1,\infty)$.

Proof. We compute

$$H'(t) = \frac{((\log t)^2 + 2\log t)(t-1)^2 - 2(t-1)t\,(\log t)^2}{(t-1)^4} = -\frac{[(t+1)\log t - 2(t-1)]\log t}{(t-1)^3}$$

for all $t > 1$. To show $H(t)$ is strictly decreasing on $(1,\infty)$, it is thus sufficient to prove that

$$\log t > \frac{2(t-1)}{t+1}$$

for all $t > 1$. This follows directly from the fact that

$$\frac{d}{dt}\left(\log t - \frac{2(t-1)}{t+1}\right) = \frac{(t-1)^2}{t(t+1)^2} > 0$$

for all $t > 1$. This proves our lemma. ☐

Before proving the monotonicity of $F^{[\kappa]}(x,a)$ in κ, we need to study

$$h^{[\kappa]}(x,a) := x^\kappa f^{[\kappa]}(x,a) = \frac{q(x,a+1)^\kappa - q(x,a)^\kappa}{q(x,a)^\kappa - q(x,a-1)^\kappa} \tag{38}$$

as a function of κ.

Proposition 4.9. *Given any positive real numbers $x > 1$ and $a \geq 1$, $h^{[\kappa]}(x,a)$ is strictly increasing and strictly log-concave as a function of $\kappa \in (1,\infty)$.*

Proof. For $a = 1$ we have $h^{[\kappa]}(x,a) = (x+1)^\kappa - 1$. It is clear that $h^{[\kappa]}(x,1)$ is strictly increasing in $\kappa \in (1,\infty)$. Since

$$\frac{\partial}{\partial\kappa}(\log h^{[\kappa]}(x,1)) = \frac{(x+1)^\kappa \log(x+1)}{(x+1)^\kappa - 1} = \left(1 + \frac{1}{(x+1)^\kappa - 1}\right)\log(x+1)$$

is strictly decreasing in $\kappa \in (1,\infty)$, we see that $h^{[\kappa]}(x,1)$ is strictly log-concave in $\kappa \in (1,\infty)$.

In what follows, we shall suppose that $a > 1$. For simplicity, let us write $A := q(x,a+1)$, $B := q(x,a)$ and $C := q(x,a-1)$. Then $A > B > C > 0$ and $AC < B^2$, the latter of which is equivalent to (37). We calculate the partial derivative $\partial h^{[\kappa]}/\partial\kappa$ to obtain

$$\begin{aligned}\frac{\partial h^{[\kappa]}}{\partial\kappa}(x,a) &= \frac{(A^\kappa \log A - B^\kappa \log B)(B^\kappa - C^\kappa) - (A^\kappa - B^\kappa)(B^\kappa \log B - C^\kappa \log C)}{(B^\kappa - C^\kappa)^2} \\ &= \frac{1}{(B^\kappa - C^\kappa)^2}\left[(AB)^\kappa \log\frac{A}{B} + (BC)^\kappa \log\frac{B}{C} - (AC)^\kappa \log\frac{A}{C}\right]\end{aligned}$$

for all $\kappa > 1$. By the weighted arithmetic mean–geometric mean inequality we have

$$\frac{\log(A/B)}{\log(A/C)}(B/C)^\kappa + \frac{\log(B/C)}{\log(A/C)}(B/A)^\kappa > \left[(B/C)^{\log(A/B)}(B/A)^{\log(B/C)}\right]^{\kappa/\log(A/C)} = 1.$$

This implies that $\partial h^{[\kappa]}/\partial\kappa > 0$ for all $\kappa > 1$. Hence $h^{[\kappa]}(x,a)$ is is strictly increasing in $\kappa \in (1,\infty)$.

Now we show that $h^{[\kappa]}(x,a)$ is strictly log-concave in $\kappa \in (1,\infty)$. Note that

$$\begin{aligned}\frac{\partial}{\partial\kappa}(\log h^{[\kappa]}(x,a)) &= \frac{1}{(A^\kappa - B^\kappa)(B^\kappa - C^\kappa)}\left[(AB)^\kappa \log\frac{A}{B} + (BC)^\kappa \log\frac{B}{C} - (AC)^\kappa \log\frac{A}{C}\right] \\ &= \frac{A^\kappa \log A - B^\kappa \log B}{A^\kappa - B^\kappa} - \frac{B^\kappa \log B - C^\kappa \log C}{B^\kappa - C^\kappa}. \end{aligned} \tag{39}$$

Since

$$\frac{\partial}{\partial\kappa}\left(\frac{A^\kappa \log A - B^\kappa \log B}{A^\kappa - B^\kappa}\right) = -\frac{A^\kappa B^\kappa (\log(A/B))^2}{(A^\kappa - B^\kappa)^2},$$

and since

$$\frac{\partial}{\partial\kappa}\left(\frac{B^\kappa \log B - C^\kappa \log C}{B^\kappa - C^\kappa}\right) = -\frac{B^\kappa C^\kappa (\log(B/C))^2}{(B^\kappa - C^\kappa)^2}$$

by symmetry, we have

$$\frac{\partial^2}{\partial\kappa^2}(\log h^{[\kappa]}(x,a)) = -\frac{A^\kappa B^\kappa (\log(A/B))^2}{(A^\kappa - B^\kappa)^2} + \frac{B^\kappa C^\kappa (\log(B/C)^2)}{(B^\kappa - C^\kappa)^2}$$

for all $\kappa > 1$. Set $r_\kappa = (A/B)^\kappa$ and $s_\kappa = (B/C)^\kappa$. Then $s_\kappa > r_\kappa > 1$, since $AC < B^2$. By Lemma 4.8 we have

$$\frac{\partial^2}{\partial\kappa^2}(\log h^{[\kappa]}(x,a)) = \frac{-H(r_\kappa) + H(s_\kappa)}{\kappa^2} < 0$$

for all $\kappa > 1$. This proves that $h^{[\kappa]}(x,a)$ is strictly log-concave in $\kappa \in (1,\infty)$ as required. □

We are now ready to show that $F^{[\kappa]}(x,a)$ is a strictly increasing function of $\kappa \in (1,\infty)$.

Proposition 4.10. *Given any positive real numbers $x > 1$ and $a \geq 1$, $F^{[\kappa]}(x,a)$ is strictly increasing in $\kappa \in (1,\infty)$.*

Proof. Fixing $x > 1$ and $a \geq 1$, we have

$$F^{[\kappa]}(x,a) = \frac{\log h^{[\kappa]}(x,a)}{\kappa \log x} - 1,$$

where $h^{[\kappa]}(x,a)$ is defined by (38). For $a = 1$ we have

$$F^{[\kappa]}(x,1) = \frac{\log((x+1)^\kappa - 1)}{\kappa \log x} - 1.$$

Since

$$\begin{aligned}\frac{\partial F^{[\kappa]}}{\partial \kappa}(x,1) &= \frac{1}{\kappa^2 \log x}\left(\frac{\kappa(x+1)^\kappa \log(x+1)}{(x+1)^\kappa - 1} - \log((x+1)^\kappa - 1)\right)\\ &= \frac{1}{\kappa^2 \log x}\left(\frac{\kappa \log(x+1)}{(x+1)^\kappa - 1} - \log(1-(x+1)^{-\kappa})\right) > 0\end{aligned}$$

for all $\kappa > 1$, it follows that $F^{[\kappa]}(x,1)$ is a strictly increasing function of $\kappa \in (1,\infty)$.

Suppose now that $a > 1$. Note that

$$\frac{\partial F^{[\kappa]}}{\partial \kappa}(x,a) = \frac{1}{\kappa^2 \log x}\left(\kappa \cdot \frac{\partial}{\partial \kappa}(\log h^{[\kappa]}(x,a)) - \log h^{[\kappa]}(x,a)\right)$$

and

$$\frac{\partial}{\partial \kappa}\left(\kappa \cdot \frac{\partial}{\partial \kappa}(\log h^{[\kappa]}(x,a)) - \log h^{[\kappa]}(x,a)\right) = \kappa \cdot \frac{\partial^2}{\partial \kappa^2}(\log h^{[\kappa]}(x,a)) < 0$$

for all $\kappa > 1$, since $h^{[\kappa]}(x,a)$ is strictly log-concave in $\kappa \in (1,\infty)$ by Proposition 4.9. Thus

$$\kappa \cdot \frac{\partial}{\partial \kappa}(\log h^{[\kappa]}(x,a)) - \log h^{[\kappa]}(x,a)$$

is strictly decreasing in $\kappa \in (1,\infty)$. It is clear that

$$\log h^{[\kappa]}(x,a) = \log \frac{q(x,a+1)^\kappa - q(x,a)^\kappa}{q(x,a)^\kappa - q(x,a-1)^\kappa} < \kappa \log \frac{q(x,a+1)}{q(x,a)},$$

since $q(x,a+1)q(x,a-1) < q(x,a)^2$. By (39) we have

$$\lim_{k \to \infty} \frac{\partial}{\partial \kappa}(\log h^{[\kappa]}(x,a)) = \log \frac{q(x,a+1)}{q(x,a)}.$$

Since $h^{[\kappa]}(x,a)$ is strictly log-concave in $\kappa \in (1,\infty)$ by Proposition 4.9, we know that $\partial \log h^{[\kappa]}/\partial \kappa$ is strictly decreasing in $\kappa \in (1,\infty)$. Hence

$$\frac{\partial}{\partial \kappa}(\log h^{[\kappa]}(x,a)) > \log \frac{q(x,a+1)}{q(x,a)}$$

for all $\kappa \in (1,\infty)$. It follows that

$$\frac{\partial h^{[\kappa]}}{\partial \kappa}(x,a) = \frac{1}{\kappa^2 \log x}\left(\kappa \cdot \frac{\partial}{\partial \kappa}(\log h^{[\kappa]}(x,a)) - \log h^{[\kappa]}(x,a)\right) > 0$$

for all $\kappa \in (1,\infty)$. Hence $F^{[\kappa]}(x,a)$ is a strictly increasing function of $\kappa \in (1,\infty)$. ☐

Proof of Theorem 4.3. The monotonicity part follows from Proposition 4.6, Proposition 4.7 and Proposition 4.10. To proceed, we observe that $F^{[\kappa]}(x,a) \to 0$ as $x \to \infty$ as a consequence of Proposition 4.6. Since $F^{[\kappa]}(x,a)$ is a strictly decreasing function of $x \in (1,\infty)$, we have $F^{[\kappa]}(x,a) > 0$ for all $x > 1$. Note that

$$\begin{aligned} f^{[\kappa]}(x,a) = \frac{(x^{a+1}-1)^\kappa - (x^a-1)^\kappa}{(x^{a+1}-x)^\kappa - (x^a-x)^\kappa} &> \frac{(x^{a+1}-1)^\kappa - (x^a-1)^\kappa + (x^a-x)^\kappa}{(x^{a+1}-x)^\kappa} \\ &= \left(\frac{x^{a+1}-1}{x(x^a-1)}\right)^\kappa + \left(\frac{x^{a-1}-1}{x^a-1}\right)^\kappa - x^{-\kappa} \\ &\to \left(1+\frac{1}{a}\right)^\kappa + \left(1-\frac{1}{a}\right)^\kappa - 1 \end{aligned}$$

as $x \to 1^+$. By Bernoulli's inequality we have

$$\left(1+\frac{1}{a}\right)^\kappa + \left(1-\frac{1}{a}\right)^\kappa - 1 > \left(1+\frac{\kappa}{a}\right) + \left(1-\frac{\kappa}{a}\right) - 1 = 1.$$

Hence $F^{[\kappa]}(x,a) \to \infty$ as $x \to 1^+$, as desired. □

We now define two partial inverses to $F^{[\kappa]}(x,a)$.

Definition 4.11. For any $x > 1$, $F^{[\kappa]}(x,\cdot) : [1,\infty) \to \left(0, \frac{\log((x+1)^\kappa-1))}{\kappa \log x} - 1\right]$ is strictly decreasing, and so it has a (strictly decreasing) inverse

$$a_x^{[\kappa]} : \left(0, \frac{\log\left((x+1)^\kappa - 1)\right)}{\kappa \log x} - 1\right] \to [1,\infty).$$

Likewise, for any $a \geq 1$, $F^{[\kappa]}(\cdot,a) : (1,\infty) \to (0,\infty)$ is strictly decreasing, and so it has a (strictly decreasing) inverse

$$x_a^{[\kappa]} : (0,\infty) \to (1,\infty).$$

By Theorem 4.3, $a_x^{[\kappa]}$ is decreasing as a function of x, $x_a^{[\kappa]}$ is decreasing as a function of a, and both $a_x^{[\kappa]}$ and $x_a^{[\kappa]}$ are increasing as functions of κ. We are interested in the equality

$$F^{[\kappa]}(x_a,a) = \epsilon,$$

so we write ϵ for the argument of $a_x^{[\kappa]}$ and of $x_a^{[\kappa]}$.

We now study $x_a^{[\kappa]}$ more carefully. To ease notation, we write x_a for $x_a^{[\kappa]}$ where no confusion arises. We typically view x_a as a function of ϵ, as noted above, but we also may consider it as a function of x via the implict function theorem and the following equality:

$$F^{[\kappa]}(x,1) = F^{[\kappa]}(x_a,a).$$

Note that x_a is *increasing* as a function of x, where it is *decreasing* as a function of ϵ.

Lemma 4.12. *For any fixed $\kappa > 1$ and $a \geq 1$, we have $x_a \sim (ax)^{1/a}$ as $\epsilon \to 0^+$ (or equivalently, as $x \to \infty$).*

Proof. Note that $x_a \to \infty$ as $\epsilon \to 0^+$. If $a \geq 1$, then

$$F^{[\kappa]}(x,a) = \frac{\log f^{[\kappa]}(x,a)}{\kappa \log x} = \frac{1}{\kappa \log x}\left(f^{[\kappa]}(x,a) - 1 + O\left((f^{[\kappa]}(x,a)-1)^2\right)\right)$$

for sufficiently large x, where $f^{[\kappa]}(x,a)$ is defined by (32). Since $(1-t)^\kappa = 1-\kappa t + O(t^2)$ for $t \in [0,1]$, it follows that

$$\begin{aligned}
f^{[\kappa]}(x,a)-1 &= \frac{(1-x^{-a-1})^\kappa - (1-x^{-a})^\kappa x^{-\kappa}}{(1-x^{-a})^\kappa - (1-x^{-a+1})^\kappa x^{-\kappa}} - 1\\
&= \frac{(1-\kappa x^{-a-1}+O(x^{-2a-2})) - (1-\kappa x^{-a}+O(x^{-2a}))x^{-\kappa}}{(1-\kappa x^{-a}+O(x^{-2a})) - (1-\kappa x^{-a+1}+O(x^{-2a+2}))x^{-\kappa}} - 1\\
&= \frac{\kappa x^{-a}(1-x^{-1}) - \kappa x^{-a-\kappa+1}(1-x^{-1}) + O(x^{-2a}+x^{-2a+2-\kappa})}{(1-\kappa x^{-a}+O(x^{-2a})) - (1-\kappa x^{-a+1}+O(x^{-2a+2}))x^{-\kappa}}\\
&= \frac{\kappa x^{-a}(1+O(x^{1-\min(\kappa,2)}))}{1+O(x^{-\min(\kappa,a)})}\\
&= \kappa x^{-a}\left(1+O\left(x^{1-\min(\kappa,2)}+x^{-\min(\kappa,a)}\right)\right)\\
&= \kappa x^{-a}\left(1+O\left(x^{1-\min(\kappa,2)}\right)\right),
\end{aligned}$$

where we have used the assumption that $\kappa > 1$ and $a \geq 1$. Thus we have

$$F^{[\kappa]}(x,a) = \frac{1+O\left(x^{1-\min(\kappa,2)}+x^{-a}\right)}{x^a \log x} = \frac{1+O\left(x^{1-\min(\kappa,2)}\right)}{x^a\log x} \sim \frac{1}{x^a \log x} \tag{40}$$

as $x \to \infty$, where all the implicit constants depend only on κ. In particular, we obtain

$$F^{[\kappa]}(x_a,a) \sim \frac{1}{x_a^a \log x_a}.$$

as $\epsilon \to 0^+$. Since $F^{[\kappa]}(x_a,a) = F^{[\kappa]}(x,1) = \epsilon$, we see that $x_a^a \log x_a \sim x\log x$ as $\epsilon \to 0^+$. The lemma follows. ☐

Now we specialize in the case $a = 2$. The following lemma provides a more precise asymptotic for x_2.

Lemma 4.13. *For any $\kappa > 1$ and $\epsilon > 0$, we have*

$$x_2 = \sqrt{2x}\left(1-\frac{\log 2}{2\log x}+O\left(\frac{1}{(\log x)^2}\right)\right) \tag{41}$$

for sufficiently large x, where the implicit constant in the error term depends only on κ.

Proof. Let $\kappa > 1$ and $\epsilon > 0$. Denote by $\xi = \xi(\kappa,\epsilon) > 1$ the unique solution to the equation $\xi^2 \log \xi = x\log x$. Since $x_2^2 \log x_2 \sim x \log x$ as $\epsilon \to 0^+$, as shown in the proof of Lemma 4.12, one may think of ξ as a proxy to x_2. Then $\xi = \sqrt{2x}(1-\eta)$, where $\eta \to 0$ as $\epsilon \to 0^+$ by Lemma 4.12. Carrying this back into the equation $\xi^2 \log\xi = x\log x$, we obtain

$$(1-2\eta+O(\eta^2))\left(1+\frac{\log 2}{\log x} - \frac{2\eta+O(\eta^2)}{\log x}\right) = 1,$$

which implies that $\eta = (\log 2 + o(1))/(2\log x)$. From this it follows that

$$\xi = \sqrt{\frac{x\log x}{\log \xi}} = \sqrt{\frac{2x\log x}{\log x + \log 2 + O(1/\log x)}} = \sqrt{2x}\left(1-\frac{\log 2}{2\log x}+O\left(\frac{1}{(\log x)^2}\right)\right). \tag{42}$$

By (40) we have

$$F^{[\kappa]}(x_2,2) = F^{[\kappa]}(x,1) = \frac{1}{x\log x} + O\left(\frac{1}{x^{\delta_\kappa}\log x}\right)$$

and

$$F^{[\kappa]}(\xi,2)=\frac{1}{\xi^2\log\xi}+O\left(\frac{1}{\xi^{1+\delta_\kappa}\log\xi}\right)=\frac{1}{x\log x}+O\left(\frac{1}{x^{(1+\delta_\kappa)/2}\log x}\right),$$

where $\delta_\kappa=\min(\kappa,2)\in(1,2]$. It follows that

$$F^{[\kappa]}(\xi,2)-F^{[\kappa]}(x_2,2)=O\left(\frac{1}{x^{(1+\delta_\kappa)/2}\log x}\right). \tag{43}$$

On the other hand, we have

$$\frac{\partial F^{[\kappa]}}{\partial x}(x,2)=\left(-\frac{F^{[\kappa]}(x,2)}{x}+\frac{1}{\kappa f^{[\kappa]}(x,2)}\cdot\frac{\partial f^{[\kappa]}}{\partial x}(x,2)\right)\frac{1}{\log x}.$$

Note that

$$\frac{1}{\kappa f^{[\kappa]}(x,2)}\cdot\frac{\partial f^{[\kappa]}}{\partial x}(x,2)=\frac{g^{[\kappa]}(x)}{[(x^3-1)^\kappa-(x^2-1)^\kappa][(x^3-x)^\kappa-(x^2-x)^\kappa]}\sim\frac{g^{[\kappa]}(x)}{x^{6\kappa}}$$

as $x\to\infty$, where

$$\begin{aligned}g^{[\kappa]}(x)=&\left[3(x^3-1)^{\kappa-1}x^2-2(x^2-1)^{\kappa-1}x\right]\left[(x^3-x)^\kappa-(x^2-x)^\kappa\right]\\&-\left[(x^3-1)^\kappa-(x^2-1)^\kappa\right]\left[(x^3-x)^{\kappa-1}(3x^2-1)-(x^2-x)^{\kappa-1}(2x-1)\right].\end{aligned}$$

But for sufficiently large x, we find that

$$\begin{aligned}3(x^3-1)^{\kappa-1}x^2-2(x^2-1)^{\kappa-1}x&=3x^{3\kappa-1}-(3\kappa-3)x^{3\kappa-4}-2x^{2\kappa-1}+O\left(x^{3\kappa-7}+x^{2\kappa-3}\right),\\(x^3-x)^\kappa-(x^2-x)^\kappa&=x^{3\kappa}-\kappa x^{3\kappa-2}-x^{2\kappa}+O\left(x^{3\kappa-4}+x^{2\kappa-1}\right),\\(x^3-1)^\kappa-(x^2-1)^\kappa&=x^{3\kappa}-\kappa x^{3\kappa-3}-x^{2\kappa}+O\left(x^{3\kappa-6}+x^{2\kappa-2}\right),\end{aligned}$$

$$(x^3-x)^{\kappa-1}(3x^2-1)-(x^2-x)^{\kappa-1}(2x-1)=3x^{3\kappa-1}-(3\kappa-2)x^{3\kappa-3}-2x^{2\kappa-1}+O\left(x^{3\kappa-5}+x^{2\kappa-2}\right).$$

It follows that

$$\begin{aligned}g^{[\kappa]}(x)&=\left(3x^{6\kappa-1}-3\kappa x^{6\kappa-3}-5x^{5\kappa-1}+O\left(x^{5\kappa-2}+x^{6\kappa-4}\right)\right)\\&\quad-\left(3x^{6\kappa-1}-(3\kappa-2)x^{6\kappa-3}-5x^{5\kappa-1}+O\left(x^{5\kappa-2}+x^{6\kappa-4}\right)\right)\\&=-2x^{6\kappa-3}+O\left(x^{5\kappa-2}+x^{6\kappa-4}\right)\end{aligned}$$

for sufficiently large x. Hence

$$\frac{1}{\kappa f^{[\kappa]}(x,2)}\cdot\frac{\partial f^{[\kappa]}}{\partial x}(x,2)\sim-2x^{-3}$$

as $x\to\infty$. Combining this with (40) we obtain

$$\frac{\partial F^{[\kappa]}}{\partial x}(x,2)=-\frac{2+o(1)}{x^3\log x}$$

as $x\to\infty$. Now the mean value theorem implies that there exists $\eta=\eta(\kappa,\epsilon)>1$ which lies between ξ and x_2, such that

$$|F^{[\kappa]}(\xi,2)-F^{[\kappa]}(x_2,2)|=\left|\frac{\partial F^{[\kappa]}}{\partial x}(\eta,2)(\xi-x_2)\right|.$$

Since $\xi\sim x_2\sim\sqrt{2x}$ as $\epsilon\to0^+$, we must have

$$\left|\frac{\partial F^{[\kappa]}}{\partial x}(\eta,2)\right|\gg\frac{1}{x^{3/2}\log x}.$$

when ϵ is sufficiently small. Together with (42) and (43) this implies that

$$x_2 = \xi + O\left(x^{1-\delta_\kappa/2}\right) = \sqrt{2x}\left(1 - \frac{\log 2}{2\log x} + O\left(\frac{1}{(\log x)^2}\right)\right)$$

for sufficiently small ϵ, since $1-\delta_\kappa/2 < 1/2$. This completes the proof of the lemma. $\square$

Remark 4.14. The quantity ξ defined by $\xi^2 \log \xi = x\log x$ may be written explicitly in terms of x by recourse to (the principal branch of) the Lambert W-function, which is defined to be the inverse of $x \mapsto xe^x$. Indeed, we have

$$\xi = e^{W(2x\log x)/2} = \sqrt{\frac{2x\log x}{W(2x\log x)}}.$$

The asymptotic (42) then follows from the asymptotic series representation for W around ∞, given by

$$W(x) = \log x - \log\log x + \sum_{m\geq 1}\sum_{n\geq 0} \frac{(-1)^n}{m!}\begin{bmatrix} n+m \\ n+1\end{bmatrix}\frac{(\log\log x)^m}{(\log x)^{m+n}}, \tag{44}$$

where $\begin{bmatrix} n \\ m\end{bmatrix}$ is an unsigned Stirling number of the first kind. The expression given in (44) converges for real $x > e$. See [13, Section 4.1.4] for more details on the asymptotics of the Lambert W-function.

Lemma 4.13 implies that $x_2 < \sqrt{2x}$ for sufficiently large x. Now we prove the following stronger result in the case $\kappa \geq 3/2$ which holds for all $x > 1$.

Lemma 4.15. *For any $\kappa \geq 3/2$ and any $\epsilon > 0$, we have $x_2 < \sqrt{2x}$.*

Proof. If $x \leq 2$, then $x_2 < x \leq \sqrt{2x}$. Suppose now that $x > 2$. Put $y = \sqrt{2x} > 2$. The inequality $x_2 < \sqrt{2x}$ is equivalent to $F^{[\kappa]}(y^2/2, 1) = F^{[\kappa]}(x_2, 2) > F^{[\kappa]}(y, 2)$. That is,

$$\frac{1}{\kappa\log(y^2/2)}\log\frac{(y^2+2)^\kappa - 2^\kappa}{y^{2\kappa}} > \frac{1}{\kappa\log y}\log\frac{(y^2+y+1)^\kappa - (y+1)^\kappa}{((y+1)^\kappa - 1)y^\kappa}.$$

Since $\log(y^2/2) < 2\log y$, it suffices to show

$$(y^2+2)^\kappa - 2^\kappa > \left(\frac{(y^2+y+1)^\kappa - (y+1)^\kappa}{(y+1)^\kappa - 1}\right)^2,$$

which can be rewritten as

$$\frac{\left(1 - \left(\frac{2}{y^2+2}\right)^\kappa\right)\left(1 - \left(\frac{1}{y+1}\right)^\kappa\right)^2}{\left(1 - \left(\frac{y+1}{y^2+y+1}\right)^\kappa\right)^2\left(1 - \frac{2y+1}{(y^2+2)(y+1)^2}\right)^\kappa} > 1. \tag{45}$$

By Lagrange's mean value theorem, there exists

$$\xi \in \left(\frac{1}{y+1}, \frac{y+1}{y^2+y+1}\right)$$

such that

$$2\log\left(1 - \left(\frac{1}{y+1}\right)^\kappa\right) - 2\log\left(1 - \left(\frac{y+1}{y^2+y+1}\right)^\kappa\right) = \frac{2\kappa\xi^{\kappa-1}y}{(1-\xi^\kappa)(y+1)(y^2+y+1)}.$$

Since

$$\frac{\xi^{\kappa-1}}{1-\xi^\kappa} > \frac{1}{(y+1)^{\kappa-1}},$$

we obtain

$$2\log\left(1-\left(\frac{1}{y+1}\right)^{k}\right)-2\log\left(1-\left(\frac{y+1}{y^2+y+1}\right)^{\kappa}\right)>\frac{2\kappa y}{(y+1)^{\kappa}(y^2+y+1)}.$$

It is clear from the inequality $\log(1-z)<-z$ for $z\in(0,1)$ that

$$-\kappa\log\left(1-\frac{2y+1}{(y^2+2)(y+1)^2}\right)>\frac{\kappa(2y+1)}{(y^2+2)(y+1)^2}.$$

From the inequality $\log(1-z)>-z-3z^2/4$ for all $z\in(0,1/3]$ it follows that

$$\log\left(1-\left(\frac{2}{y^2+2}\right)^{\kappa}\right)>-\left(\frac{2}{y^2+2}\right)^{\kappa}-\frac{3}{4}\left(\frac{2}{y^2+2}\right)^{2\kappa}.$$

Thus the natural logarithm of the left-hand side of (45) is greater than

$$\frac{2\kappa y}{(y+1)^{\kappa}(y^2+y+1)}+\frac{\kappa(2y+1)}{(y^2+2)(y+1)^2}-\left(\frac{2}{y^2+2}\right)^{\kappa}-\frac{3}{4}\left(\frac{2}{y^2+2}\right)^{2\kappa}.$$

We have to show that the above expression is greater than 0, or equivalently,

$$\frac{2\kappa y}{y^2+y+1}+\frac{\kappa(2y+1)(y+1)^{\kappa-2}}{y^2+2}-\left(\frac{2(y+1)}{y^2+2}\right)^{\kappa}-\frac{3}{4}\left(\frac{2\sqrt{y+1}}{y^2+2}\right)^{2\kappa}>0.$$

Since $y^2+2>2(y+1)$ and $y^2+2>2\sqrt{y+1}$ for any $y>2$, each term on the left-hand side with the sign attached is an increasing function of $\kappa\in[3/2,\infty)$. Hence it suffices to prove

$$\frac{3y}{y^2+y+1}+\frac{3(2y+1)}{2(y^2+2)\sqrt{y+1}}-\left(\frac{2(y+1)}{y^2+2}\right)^{3/2}-\frac{3}{4}\left(\frac{2\sqrt{y+1}}{y^2+2}\right)^{3}>0 \tag{46}$$

for all $y>2$. But

$$\begin{aligned}\frac{2y+1}{(y^2+2)\sqrt{y+1}}-\left(\frac{2\sqrt{y+1}}{y^2+2}\right)^{3}&=\frac{(2y+1)(y^2+2)^2-8(y+1)^2}{(y^2+2)^3\sqrt{y+1}}\\&>4\cdot\frac{(y^2+2)^2-(2y+2)^2}{(y^2+2)^3\sqrt{y+1}}>0,\end{aligned}$$

where the last inequality holds because $2y+1>4$. Using the facts that $3y>2(y+1)$ and that $2y+1>y+1$, the left-hand side of (46) is

$$\begin{aligned}&>\frac{3y}{y^2+y+1}+\frac{3(2y+1)}{4(y^2+2)\sqrt{y+1}}-\left(\frac{2(y+1)}{y^2+2}\right)^{3/2}\\&>\frac{2(y+1)}{y^2+y+1}+\frac{3\sqrt{y+1}}{4(y^2+2)}-\left(\frac{2(y+1)}{y^2+2}\right)^{3/2}\\&=\frac{(y+1)^{3/2}}{4(y^2+2)^{3/2}}\left(\frac{8(y^2+2)^{3/2}}{(y^2+y+1)\sqrt{y+1}}+\frac{3\sqrt{y^2+2}}{y+1}-8\sqrt{2}\right).\end{aligned}\tag{47}$$

To prove (46), it suffices to show that the factor in the parentheses in (47) is positive. Since

$$\frac{d}{dy}\left(\frac{y^2+2}{(y+1)^2}\right)=\frac{2(y-2)}{(y+1)^3}>0$$

for all $y>2$, it follows that

$$\frac{3\sqrt{y^2+2}}{y+1}=3\sqrt{\frac{y^2+2}{(y+1)^2}}$$

is strictly increasing on $[2,\infty)$. Similarly, we have

$$\frac{d}{dy}\left(\frac{2(y^2+2)^{3/2}}{(y^2+y+1)\sqrt{y+1}}\right)=\frac{\sqrt{y^2+2}}{(y^2+y+1)^2(y+1)^{3/2}}\left(y^4+5y^3-y^2-8y-6\right)>0,$$

since

$$y^4+5y^3-y^2-8y-6=(y^2(y^2-1)-6)+y(5y^2-8)>0$$

for all $y>2$. Thus

$$\frac{8(y^2+2)^{3/2}}{(y^2+y+1)\sqrt{y+1}}$$

is also strictly increasing on $[2,\infty)$. It follows that the factor in the parentheses in (47) is greater than

$$\frac{48\sqrt{2}}{7}+\sqrt{6}-8\sqrt{2}>\frac{48\sqrt{2}}{7}+\frac{3\sqrt{2}}{2}-8\sqrt{2}=\frac{5\sqrt{2}}{14}>0$$

for all $y>2$. This completes the proof that (47) is positive and hence that of the lemma. ☐

Remark 4.16. It seems that Lemma 4.15 holds for all $\kappa>1$. This would follow if one could show that the function

$$[(x^2+2)^\kappa-2^\kappa][(x+1)^\kappa-1]^2-[(x^2+x+1)^\kappa-(x+1)^\kappa]^2$$

is strictly increasing in $x\in[0,\infty)$ and $\kappa\in[1,\infty)$. In fact, monotonicity in either of the two variable suffices.

If $\kappa=2$, we can strengthen Lemma 4.13 substantially.

Lemma 4.17. *We have*

$$\sqrt{2x}\left(1-\frac{\log 2}{2\log x}\right)<x_2^{[2]} \tag{48}$$

whenever $x\geq 2^{15}$.

Proof. We follow the proof of [3, Lemma 6.17]. Let

$$y\coloneqq\sqrt{2x}\left(1-\frac{\log 2}{2\log x}\right), \tag{49}$$

so (48) is equivalent to

$$\frac{\log\left(1+\frac{2}{x}\right)}{2\log x}=F^{[2]}(x,1)=F^{[2]}(x_2,2)<F^{[2]}(y,2)=\frac{\log\left(1+\frac{2}{y^2+2y}\right)}{2\log y}. \tag{50}$$

Recall that

$$\frac{1}{t+1}<\log\left(1+\frac{1}{t}\right)<\frac{1}{t} \tag{51}$$

whenever $t>1$. Using (51), we find (50) follows from

$$2\left(y^2+2y+2\right)\frac{\log y}{\log x}<2x. \tag{52}$$

For ease of notation, we write $u\coloneqq\frac{\log 2}{\log x}$, so $y=\sqrt{2x}\,(1-u/2)$. We have

$$2\frac{\log y}{\log x}=\frac{\log 2+\log x+2\log\left(1-\frac{\log 2}{2\log x}\right)}{\log x}<1+\frac{\log 2}{\log x}-\frac{\log 2}{(\log x)^2}=1+u-\frac{u^2}{\log 2}, \tag{53}$$

by (49) and because $\log(1-t)<-t$ whenever $0<t<1$.

Note that $u \leq 1/15$ by the hypothesis $x \geq 2^{15}$. We now estimate the left-hand side of (52) in two pieces. First, we have

$$2y^2 \frac{\log y}{\log x} < 2x \left(1 - \frac{u}{2}\right)^2 \left(1 + u - \frac{u^2}{\log 2}\right) < 2x \left(1 - 2.0804u^2\right),$$

where the first inequality follows from (53) and the last inequality follows because

$$\begin{aligned} \left(1 - \frac{u}{2}\right)^2 \left(1 + u - \frac{u^2}{\log 2}\right) &= 1 - \left(3 + \frac{4}{\log 2} - \left(1 + \frac{4}{\log 2}\right) u + \frac{u^2}{\log 2}\right) \frac{u^2}{4} \\ &\leq 1 - \left(3 + \frac{4}{\log 2} - \left(1 + \frac{4}{\log 2}\right) u + \frac{u^2}{\log 2}\right) \Bigg|_{u=\frac{1}{15}} \frac{u^2}{4} \\ &< 1 - 2.0804u^2. \end{aligned}$$

Second, we have

$$\begin{aligned} 4\left(y+1\right) \frac{\log y}{\log x} &< 2\left(y+1\right) \left(1 + u - \frac{u^2}{\log 2}\right) \\ &\leq \left(2\left(1 - \frac{u}{2}\right)\left(1 + u - \frac{u^2}{\log 2}\right)\sqrt{2x} + 2\left(1 + u - \frac{u^2}{\log 2}\right)\right)\Bigg|_{u=\frac{1}{15}} \\ &\leq 2.059\sqrt{2x}, \end{aligned}$$

where the first inequality again follows from (53). Therefore,

$$2\left(y^2 + 2y + 2\right) \frac{\log y}{\log x} \leq 2x \left(1 - 2.0804u^2\right) + 2.059\sqrt{2x} < 2x,$$

where the final inequality follows because

$$-4.1608u^2 x + 2.059\sqrt{2x} < -67 < 0$$

for $x \geq 2^{15}$. This completes the proof. □

4.2. **Understanding the κ-colossally abundant numbers.**

With the notation and results we have developed, we can now understand κ-colossally abundant numbers using $F^{[\kappa]}$.

Proposition 4.18. *If N is κ-colossally abundant for ϵ, then for every prime p, we have*

$$F^{[\kappa]}(p, v_p(N) + 1) \leq \epsilon \leq F^{[\kappa]}(p, v_p(N)), \tag{54}$$

where v_p is the p-adic valuation on $\mathbb{Q}$. Conversely, if N is a positive integer and $\epsilon > 0$ is a real number, and (54) holds for every prime p, then N is κ-colossally abundant for ϵ.

Proof. Let $\kappa > 1$ and $\epsilon > 0$ be given, and let p be a prime. From Definition 4.2 and (29), we conclude

$$f_\epsilon^{[\kappa]}(p,1) f_\epsilon^{[\kappa]}(p,2) \cdots f_\epsilon^{[\kappa]}(p,a) = \rho_\epsilon^{[\kappa]}(p^a) = \frac{\sigma^{[\kappa]}(p^a)}{p^{a\kappa(1+\epsilon)}}.$$

Now $\frac{\sigma^{[\kappa]}(p^a)}{p^{a\kappa(1+\epsilon)}} \geq \frac{\sigma^{[\kappa]}(p^{a-1})}{p^{(a-1)\kappa(1+\epsilon)}}$ if and only if $f_\epsilon^{[\kappa]}(p,a) \geq 1$. By Proposition 4.7, $f_\epsilon^{[\kappa]}(p,a)$ is a strictly decreasing function of $a \geq 1$, and $\lim_{a\to\infty} f_\epsilon^{[\kappa]}(p,a) = p^{-\kappa\epsilon} < 1$. Write $X_p \coloneqq \{\, p^a \;:\; a \in \mathbb{Z}_{\geq 0} \,\}$ for ease of notation. If $f_\epsilon^{[\kappa]}(p,a) < 1$ for all $a \geq 1$, then $\rho_\epsilon^{[\kappa]}|_{X_p}$ is maximized at $p^a = 1$; otherwise, it is maximized when $f_\epsilon^{[\kappa]}(p,a) \geq 1$ and $f_\epsilon^{[\kappa]}(p,a+1) \leq 1$. In particular, if $f_\epsilon^{[\kappa]}(p,a) > 1$ for some a and $f_\epsilon^{[\kappa]}(p,a) \neq 1$ for all a, then $\rho_\epsilon^{[\kappa]}|_{X_p}$ is maximized when a is chosen so $f_\epsilon^{[\kappa]}(p,a) > 1$ and $f_\epsilon^{[\kappa]}(p,a+1) < 1$; if $f_\epsilon^{[\kappa]}(p,a) = 1$ for some $a \geq 1$, then p^{a-1} and

p^a are the unique global maxima of $\rho_\epsilon^{[\kappa]}|_{X_p}$, and of course $\rho_\epsilon^{[\kappa]}|_{X_p}(p^{a-1}) = \rho_\epsilon^{[\kappa]}|_{X_p}(p^a)$. With the convention $F^{[\kappa]}(p,0) := \infty$, we have shown $\rho_\epsilon^{[\kappa]}|_{X_p}$ is maximized at p^a if

$$F^{[\kappa]}(p,a+1) \leq \epsilon \leq F^{[\kappa]}(p,a).$$

Observe also that $\lim_{p\to\infty} f_\epsilon^{[\kappa]}(p,1) = 0$, so $\rho_\epsilon^{[\kappa]}|_{X_p}$ is maximized at $p^a = 1$ for all p sufficiently large.

Now as $\rho_\epsilon^{[\kappa]}$ is multiplicative, every κ-colossally abundant number N for ϵ is of the form $N = \prod_{p \text{ prime}} p^{a_p}$, where p^{a_p} maximizes $\rho_\epsilon^{[\kappa]}|_{X_p}$, and every number obtained in this fashion is κ-colossally abundant number for ϵ. □

The proof of Proposition 4.18 above shows that the triples (p, a, ϵ) for which $f_\epsilon^{[\kappa]}(p,a) = 1$ are especially important to our understanding of κ-colossally abundant numbers, so we make the following definition.

Definition 4.19. Let

$$E_p^{[\kappa]} := \{\, F^{[\kappa]}(p,j) \ : \ j \in \mathbb{Z}_{>0} \,\}$$

for p prime, and define

$$E^{[\kappa]} := \bigcup_{p \text{ prime}} E_p^{[\kappa]} \cup \{\infty\}.$$

Write $E^{[\kappa]} = \{\, \epsilon_i^{[\kappa]} \,\}_{i=0}^{\infty}$, where $\infty = \epsilon_0^{[\kappa]} > \epsilon_1^{[\kappa]} > \epsilon_2^{[\kappa]} > \cdots > 0$, and $\lim_{i\to\infty} \epsilon_i^{[\kappa]} = 0$. For $0 < \epsilon < \infty$, if $\epsilon \in E^{[\kappa]}$, we say ϵ is a κ-critical value. Otherwise, ϵ is a κ-noncritical value.

Thus, a real number ϵ is a critical value precisely if $f_\epsilon^{[\kappa]}(p,a) = 1$ for some prime p and some integer a.

Definition 4.20. For each $\epsilon > 0$, we define $N^{[\kappa]}(\epsilon)$ to be the largest κ-colossally abundant number for ϵ.

Example 4.21. Let $\kappa = 2.956801214357021\ldots$, and note that

$$F^{[\kappa]}(2,5) = F^{[\kappa]}(5,2) = \epsilon_{13}^{[\kappa]} = 0.019785233524272305\ldots.$$

We have

$$N^{[\kappa]}\left(\epsilon_{13}^{[\kappa]}\right) = 2^5 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 = 2 \cdot 5 \cdot N^{[\kappa]}\left(\epsilon_{12}^{[\kappa]}\right).$$

However, both $2 \cdot N^{[\kappa]}\left(\epsilon_{12}^{[\kappa]}\right)$ and $5 \cdot N^{[\kappa]}\left(\epsilon_{12}^{[\kappa]}\right)$ are also κ-colossally abundant for $\epsilon_{13}^{[\kappa]}$.

More broadly, if $E_p^{[\kappa]} \cap E_q^{[\kappa]} \neq \emptyset$ and $\epsilon = \epsilon_i \in E_p^{[\kappa]} \cap E_q^{[\kappa]}$, then $N^{[\kappa]}(\epsilon_i)$ is a (likely trivial) multiple of $pqN^{[\kappa]}(\epsilon_{i-1})$, but $pN^{[\kappa]}(\epsilon_{i-1})$ and $qN^{[\kappa]}(\epsilon_{i-1})$ are also κ-colossally abundant numbers for ϵ_i. Thus $\{\, N^{[\kappa]}(\epsilon) \,\}_{\epsilon>0}$ may be a proper subset of

$$\{\, N \in \mathbb{Z}_{>0} \ : \ N \text{ is } \kappa\text{-colossally abundant} \,\}.$$

Theorem 4.22. *For all $\epsilon > 0$, we have*

$$N^{[\kappa]}(\epsilon) = \prod_{p \text{ prime}} p^{\alpha_p^{[\kappa]}(\epsilon)},$$

where

$$\alpha_p^{[\kappa]}(\epsilon) := \left\lfloor a_p^{[\kappa]}(\epsilon) \right\rfloor.$$

If ϵ is κ-noncritical, then $N^{[\kappa]}(\epsilon)$ is the only κ-colossally abundant number for ϵ.

Proof. Fix a colossally abundant number N for ϵ, and let p be prime. Let a be the p-adic valuation of N. By Proposition 4.18, we see

$$F^{[\kappa]}(p,a+1) \leq \epsilon \leq F^{[\kappa]}(p,a).$$

On the other hand, we know from Theorem 4.3 that $F^{[\kappa]}(p,a)$ is strictly decreasing in a, so by the definition of $a_p^{[\kappa]}$, the above inequality is equivalent to

$$a \leq a_p^{[\kappa]}(\epsilon) \leq a+1,$$

or equivalently,

$$a_p^{[\kappa]}(\epsilon) - 1 \leq a \leq a_p^{[\kappa]}(\epsilon). \tag{55}$$

Now if $\epsilon \notin E_p^{[\kappa]}$, then both inequalities in (55) are strict, and so $a = \left\lfloor a_p^{[\kappa]}(\epsilon) \right\rfloor$. In particular, if ϵ is κ-noncritical then $N^{[\kappa]}(\epsilon)$ is the only κ-colossally abundant number for ϵ. Also, if $\epsilon \in E_p^{[\kappa]}$, then $a_p^{[\kappa]}(\epsilon) \in \mathbb{Z}$, and (55) implies $a \in \{ a_p^{[\kappa]}(\epsilon) - 1, a_p^{[\kappa]}(\epsilon) \}$. But as $N^{[\kappa]}(\epsilon)$ is maximal by assumption, we again see that $a = \left\lfloor a_p^{[\kappa]}(\epsilon) \right\rfloor = a_p^{[\kappa]}(\epsilon)$. ☐

Corollary 4.23. *If $\kappa < \kappa'$, then $N^{[\kappa]}(\epsilon)$ divides $N^{[\kappa']}(\epsilon)$.*

Proof. Theorem 4.3 tells us $F^{[\kappa]}$ is increasing in κ, so likewise $a_p^{[\kappa]}$ is increasing in κ for each prime p. The claim is now immediate. ☐

We emphasize that Corollary 4.23 does *not* imply that $N^{[\kappa]}\left(\epsilon_i^{[\kappa]}\right)$ divides $N^{[\kappa']}\left(\epsilon_i^{[\kappa']}\right)$.

Example 4.24. We have $N^{[2]}\left(\epsilon_6^{[2]}\right) = 2^3 \cdot 3 \cdot 5 \cdot 7$, but $N^{[3]}\left(\epsilon_6^{[3]}\right) = 2^3 \cdot 3^2 \cdot 5$. In this case, we have $\epsilon_6^{[2]} = \frac{\log(51/5)}{2\log 3} - 1 \approx 0.05696$, and $\epsilon_6^{[3]} = \frac{\log(511)}{3\log 7} - 1 \approx 0.06829$.

If $E_p^{[\kappa]} \cap E_q^{[\kappa]} = E_p^{[\kappa']} \cap E_q^{[\kappa']} = \emptyset$ for all primes $p \neq q$, then $N^{[\kappa]}\left(\epsilon_i^{[\kappa]}\right)$ and $N^{[\kappa']}\left(\epsilon_i^{[\kappa']}\right)$ must both have i prime factors, counting multiplicity, so one divides the other exactly if one equals the other.

Theorem 4.22 can be reformulated in terms of $x_a^{[\kappa]}$ instead of $a_x^{[\kappa]}$ as follows:

$$N^{[\kappa]}(\epsilon) = \prod_{\ell \geq 1} \prod_{x_{\ell+1} < p \leq x_\ell} p^\ell, \tag{56}$$

where p varies over the set of primes $\{2,3,5,7,\dots\}$, and $x_\ell \coloneqq x_\ell^{[\kappa]}(\epsilon)$.

Theorem 4.25. *The function $N^{[\kappa]}(\epsilon)$ is constant on each half-open interval $[\epsilon_i^{[\kappa]}, \epsilon_{i-1}^{[\kappa]})$. These constant values are distinct and increasing. The number $N^{[\kappa]}\left(\epsilon_i^{[\kappa]}\right)$ is also κ-colossally abundant for $\epsilon_{i+1}^{[\kappa]}$.*

Proof. Note

$$\{ \epsilon_i^{[\kappa]} \}_{i=1}^{\infty} = \{ \epsilon \in (0,\infty) \ : \ a_p^{[\kappa]}(\epsilon) \in \mathbb{Z} \text{ for some prime } p \}.$$

Thus if $\epsilon, \epsilon' \in (\epsilon_i^{[\kappa]}, \epsilon_{i-1}^{[\kappa]})$, then for each prime p and each integer a, we have the inequality $a_p^{[\kappa]}(\epsilon) - 1 \leq a \leq a_p^{[\kappa]}(\epsilon)$ if and only if $a_p^{[\kappa]}(\epsilon') - 1 \leq a \leq a_p^{[\kappa]}(\epsilon')$. We have shown that $N^{[\kappa]}(\epsilon)$ is constant for $\epsilon \in (\epsilon_i^{[\kappa]}, \epsilon_{i-1}^{[\kappa]})$. But the function $\alpha_p^{[\kappa]}(\cdot) = \left\lfloor a_p^{[\kappa]}(\cdot) \right\rfloor$ is a composition of right-continuous functions, and so is right-continuous; thus

$$\alpha_p^{[\kappa]}(\epsilon_i^{[\kappa]}) = \lim_{\epsilon \to (\epsilon_i^{[\kappa]})^+} \alpha_p^{[\kappa]}(\epsilon);$$

so in fact $N^{[\kappa]}(\epsilon)$ is constant for $\epsilon \in [\epsilon_i^{[\kappa]}, \epsilon_{i-1}^{[\kappa]})$. Moreover, if $\epsilon < \epsilon_i^{[\kappa]} < \epsilon'$, then for some prime p (not necessarily unique) and some integer $a > 0$ we have

$$a_p^{[\kappa]}(\epsilon) < a < a_p^{[\kappa]}(\epsilon'),$$

so $\alpha_p^{[\kappa]}(\epsilon) \leq a - 1 < a = \alpha_p^{[\kappa]}(\epsilon')$. This proves the second claim.

It remains to show that $N^{[\kappa]}\left(\epsilon_i^{[\kappa]}\right)$ is κ-colossally abundant for $\epsilon_{i+1}^{[\kappa]}$. For p prime, if $\epsilon_{i+1}^{[\kappa]} \notin E_p^{[\kappa]}$ then $\alpha_p^{[\kappa]}(\epsilon_i^{[\kappa]}) = \alpha_p^{[\kappa]}(\epsilon_{i+1}^{[\kappa]})$. If $\epsilon_{i+1}^{[\kappa]} \in E_p^{[\kappa]}$ then $\alpha_p^{[\kappa]}(\epsilon_i^{[\kappa]}) + 1 = \alpha_p^{[\kappa]}(\epsilon_{i+1}^{[\kappa]})$, but by Proposition 4.18, replacing $\alpha_p^{[\kappa]}(\epsilon_{i+1}^{[\kappa]})$ with $\alpha_p^{[\kappa]}(\epsilon_{i+1}^{[\kappa]}) - 1 = \alpha_p^{[\kappa]}(\epsilon_i^{[\kappa]})$ also yields a nonmaximal κ-colossally abundant number for $\epsilon_{i+1}^{[\kappa]}$. Thus $N^{[\kappa]}\left(\epsilon_i^{[\kappa]}\right)$ is κ-colossally abundant for $\epsilon_{i+1}^{[\kappa]}$ as desired. ☐

Theorem 4.25 shows that a κ-colossally abundant number of the form $N^{[\kappa]}(\epsilon)$ is in fact of the form $N^{[\kappa]}\left(\epsilon_i^{[\kappa]}\right)$ for some $i \geq 1$. It is therefore natural to index these κ-colossally abundant numbers by an integral parameter instead of a real parameter.

Definition 4.26. For $\kappa > 1$ given and for $i \geq 1$, we define

$$N_i^{[\kappa]} := N^{[\kappa]}\left(\epsilon_i^{[\kappa]}\right).$$

Corollary 4.27. *For any $\kappa > 1$, there are infinitely many κ-colossally abundant numbers.*

Proof. We observe $\{ N_i^{[\kappa]} \}_{i=1}^{\infty}$ is a strictly increasing sequence of κ-colossally abundant numbers. ☐

The following definition is motivated by Theorem 1.4.

Definition 4.28. For κ a positive real number, we define

$$G^{[\kappa]}(n) := \frac{\zeta(\kappa)\sigma^{[\kappa]}(n)}{(e^{\gamma} n \log\log n)^{\kappa}}.$$

Theorem 4.29. *If $N_i^{[\kappa]} \leq n \leq N_{i+1}^{[\kappa]}$, then*

$$G^{[\kappa]}(n) \leq \max\left(G^{[\kappa]}\left(N_i^{[\kappa]}\right), G^{[\kappa]}\left(N_{i+1}^{[\kappa]}\right)\right).$$

Proof. Write $\epsilon := \epsilon_{i+1}$, $N_i := N_i^{[\kappa]}$, and $N_{i+1} := N_{i+1}^{[\kappa]}$ to clean up notation. By Theorem 4.25, N_i and N_{i+1} are both critical points of the function

$$n \mapsto \frac{\sigma^{[\kappa]}(n)}{n^{\kappa(1+\epsilon)}}$$

that yield the same (global) maximum. The function $f(x) := \epsilon x - \log\log x$ is concave upwards for $x > 1$, and so for $1 < a \leq \xi \leq b$, we have $f(\xi) \leq \max(f(a), f(b))$. In particular, we see

$$\epsilon \log n - \log\log\log n \leq \max\left(\epsilon \log N_i - \log\log\log N_i, \epsilon \log N_{i+1} - \log\log\log N_{i+1}\right);$$

multiplying through by $\kappa > 0$, taking exponentials, and multiplying by $\zeta(\kappa)e^{-\kappa\gamma}$, we see

$$\frac{\zeta(\kappa) n^{\kappa\epsilon}}{(e^{\gamma} \log\log n)^{\kappa}} \leq \max\left(\frac{\zeta(\kappa) N_i^{\kappa\epsilon}}{(e^{\gamma} \log\log N_i)^{\kappa}}, \frac{\zeta(\kappa) N_{i+1}^{\kappa\epsilon}}{(e^{\gamma} \log\log N_{i+1})^{\kappa}}\right).$$

On the other hand,

$$\frac{\sigma^{[\kappa]}(n)}{n^{\kappa(1+\epsilon)}} \leq \frac{\sigma^{[\kappa]}(N_i)}{N_i^{\kappa(1+\epsilon)}} = \frac{\sigma^{[\kappa]}(N_{i+1})}{N_{i+1}^{\kappa(1+\epsilon)}},$$

and multiplying these two inequalities together yields the desired result. ☐

We close this section with a lemma that relates $x_1^{[\kappa]}(\epsilon)$ to $N^{[\kappa]}(\epsilon)$, conditional on the Riemann hypothesis.

Lemma 4.30. *Fix $\kappa \geq 3/2$, and assume the Riemann hypothesis holds. If $x = x_1^{[\kappa]}(\epsilon)$ and $N = N^{[\kappa]}(\epsilon)$, then we have*

$$\log\log N > \log\theta(x)\exp\left(\frac{0.977\sqrt{2}}{\sqrt{x}\log x} + O\left(\frac{1}{\sqrt{x}\,(\log x)^2}\right)\right) \tag{57}$$

for sufficiently large x. Moreover, if $\kappa = 2$ then

$$\log\log N > \log\theta(x)\exp\left(\frac{0.977\sqrt{2}}{\sqrt{x}\log x}\left(1 - \frac{\log 2}{2\log x}\right)\right) \tag{58}$$

for $x \geq 2^{15}$.

Proof. We follow the proof of [3, Lemma 7.10].

Suppose $x \geq 2^{15}$. As usual, we write x_ℓ for $x_\ell^{[\kappa]}(\epsilon)$. By (56), we see $\log N = \sum_{\ell \geq 1}\theta(x_\ell) \geq \theta(x) + \theta(x_2)$. Now by Lemma 2.7, we see

$$\log N > \theta(x)\left(1 + \frac{0.985x_2}{1.000081x}\right).$$

Taking logarithms again, recalling that $\log(1+x) > \frac{x}{x+1}$ for $x > 0$, and using Lemma Lemma 2.7 and Lemma 4.15, we see

$$\begin{aligned}
\log\log N &> \log\theta(x) + \log\left(1 + \frac{0.985x_2}{1.000081x}\right)\\
&> \log\theta(x) + \frac{0.985x_2}{1.000081x + 0.985x_2}\\
&> \log\theta(x) + \frac{0.985x_2}{1.000081x + 0.985\sqrt{2x}}\\
&> \log\theta(x) + \frac{0.977399x_2}{x}\\
&> \log\theta(x)\left(1 + \frac{0.977399x_2}{x\log(1.000081x)}\right)\\
&> \log\theta(x)\left(1 + \frac{0.977391x_2}{x\log x}\right).
\end{aligned}$$

Now for real $m, b > 0$, and $t \in [0, b]$, if $m \leq \frac{\log(1+b)}{b}$ then $1 + t \geq e^{mt}$. But by Lemma 4.15, $x_2 < \sqrt{2x}$, so for $x \geq 2^{15}$ we have

$$\frac{0.977391x_2}{x\log x} < \frac{0.977391\sqrt{2}}{\sqrt{x}\log x} \leq \frac{0.977391}{2^7\log 2^{15}} < b := 0.000735.$$

Taking $m := \frac{\log(1+b)}{b} > 0.9996$ and observing $0.977391m > 0.977$, we see

$$\log\log N > \log\theta(x)\exp\left(\frac{0.977x_2}{x\log x}\right). \tag{59}$$

Substituting (41) into (59) yields (57), and substituting (48) into (59) yields (58). ☐

5. An analogue to Robin's theorem

In 1984, Robin proved the following converse of Ramanujan's result.

Theorem 5.1 ([19, p. 204], Robin's Theorem)**.** *If the Riemann hypothesis fails, let B be the supremum of the real parts of the nontrivial zeros of $\zeta(s)$, and let $b \in (1-B, 1/2)$. There is a constant $c = c(b) > 0$ such that the inequality*

$$\sigma(n) > e^{\gamma} n \log\log n \left(1 + \frac{c}{(\log n)^b}\right)$$

holds for infinitely many n.

Our goal in this section is to prove the following result, which is a corollary to Theorem 5.1.

Theorem 5.2. *Let $\kappa > 1$ be a real number. If the Riemann hypothesis fails, let B be the supremum of the real parts of the nontrivial zeros of $\zeta(s)$, and let $b \in (1-B, 1/2)$. There is a constant $c = c(b) > 0$ independent of κ such that the inequality*

$$\sigma^{[\kappa]}(n) > \frac{(e^{\gamma} n \log\log n)^{\kappa}}{\zeta(\kappa)} \left(1 + \frac{c}{(\log n)^b}\right)^{\kappa}$$

holds for infinitely many n.

Theorem 5.2 gives the implication (2) $\Longrightarrow$ (1) of Theorem 1.5. Our proof of Theorem 5.2 has two key components: the first is Theorem 5.1 itself, and the second is the following lemma.

Lemma 5.3. *Fix $n \in \mathbb{Z}_{>0}$. The function*

$$\kappa \mapsto \left(\zeta(\kappa)\sigma^{[\kappa]}(n)\right)^{1/\kappa}.$$

is smooth and monotonically decreasing in $\kappa \in (1, \infty)$; if $n > 1$, this function is strictly decreasing in κ.

The proof of Lemma 5.3 itself requires the following simple result.

Lemma 5.4. *If $v > u > 1$, then*

$$H(x; u, v) := \left(\frac{v^x - 1}{u^x - 1}\right)^{1/x}$$

is a strictly decreasing function of $x \in (0, \infty)$.

Proof. We show that the monotonicity of $H(x; u, v)$ follows quickly from that of $F(x, a)$, which we proved in Proposition 4.4. To this end, we consider

$$\log H(x; u, v) = \frac{1}{x} \log\left(\frac{v^x - 1}{u^x - 1}\right).$$

Suppose first that $u < v \leq u^2$. Observe

$$\log H(x; u, v) = (F((v/u)^x, (\log u)/\log(v/u)) + 1)\log(v/u).$$

Since for fixed $a > 1$, $F(x, a)$ is strictly decreasing in $x > 1$, we see that $\log H(x; u, v)$ is strictly decreasing in $x > 0$. In the general case, we may assume $u^{2^k} < v \leq u^{2^{k+1}}$, where $k \geq 0$ is some integer. Then

$$\log H(x; u, v) = \sum_{i=0}^{k-1} \log H(x; u^{2^i}, u^{2^{i+1}}) + \log H(x; u^{2^k}, v).$$

By the special case that we just handled, each summand is strictly decreasing in $x > 0$. Therefore, $\log H(x; u, v)$ is strictly decreasing in $x > 0$, and so is $H(x; u, v)$. □

Proof of Lemma 5.3. Suppose $\kappa' > \kappa > 1$. We will prove the function

$$\left(\frac{\sigma^{[\kappa]}(p^a)}{1-p^{-\kappa}}\right)^{1/\kappa}$$

is decreasing in κ for each prime p and each integer $a \geq 0$. Taking a product over all primes p, with $a = a(p) = 0$ for almost all p, will prove the result.

Suppose first that $a = 0$. In this case,

$$\left(\frac{\sigma^{[\kappa]}(p^a)}{1-p^{-\kappa}}\right)^{1/\kappa} = \frac{1}{(1-p^{-\kappa})^{1/\kappa}},$$

so it suffices to prove that $(1-p^{-\kappa})^{1/\kappa}$ is increasing. But if $\kappa' > \kappa$ then

$$\left(1-p^{-\kappa'}\right)^{1/\kappa'} > \left(1-p^{-\kappa}\right)^{1/\kappa'} > \left(1-p^{-\kappa}\right)^{1/\kappa},$$

so $(1-p^{-\kappa})^{1/\kappa}$ is increasing as desired.

Suppose now that $a > 0$. We recall that $\sigma(p^a) = p\sigma(p^{a-1}) + 1$, so

$$\begin{aligned}\left(\frac{\sigma^{[\kappa]}(p^a)}{1-p^{-\kappa}}\right)^{1/\kappa} &= \left(\frac{\sigma(p^a)^\kappa - \sigma(p^{a-1})^\kappa}{1-p^{-\kappa}}\right)^{1/\kappa} \\ &= p\sigma(p^{a-1})\left(\frac{(p+\sigma(p^{a-1})^{-1})^\kappa - 1}{p^\kappa - 1}\right)^{1/\kappa}.\end{aligned}$$

Taking $u = p$ and $v = p + \sigma(p^{a-1})^{-1}$ in Lemma 5.4, we see

$$\left(\frac{(p+\sigma(p^{a-1})^{-1})^\kappa - 1}{p^\kappa - 1}\right)^{1/\kappa}$$

is a decreasing function of κ, and the result follows. □

The proof of Theorem 5.2 is now straightforward.

Proof of Theorem 5.2. For any $n > 1$, we have

$$\lim_{\kappa\to\infty}\left(\zeta(\kappa)\sigma^{[\kappa]}(n)\right)^{1/\kappa} = \sigma(n).$$

Thus by Lemma 5.3, for fixed $\kappa > 1$ we have

$$\left(\zeta(\kappa)\sigma^{[\kappa]}(n)\right)^{1/\kappa} > \sigma(n).$$

Suppose now that the Riemann hypothesis fails, and let b and c be as in Theorem 5.1. There are infinitely many n for which

$$\left(\zeta(\kappa)\sigma^{[\kappa]}(n)\right)^{1/\kappa} > \sigma(n) > e^\gamma n\log\log n\left(1+\frac{c}{(\log n)^b}\right).$$

Rearranging, we obtain

$$\sigma^{[\kappa]}(n) > \frac{\sigma(n)^\kappa}{\zeta(\kappa)} > \frac{(e^\gamma n\log\log n)^\kappa}{\zeta(\kappa)}\left(1+\frac{c}{(\log n)^b}\right)^\kappa,$$

which is what we desired to show. □

Remark 5.5. Robin's proof of Theorem 5.1 depends on the inequality

$$x_\ell > x^{1/\ell}. \tag{60}$$

For fixed $\kappa > 1$ and $\ell \geq 2$, Lemma 4.12 implies that (60) holds for sufficiently large x, but we have been unable to establish this inequality uniformly in ℓ, nor have we been able to adapt Robin's original argument to route around this claim. The difficulty is due to $F^{[\kappa]}(x, a)$ being more complicated than Robin's $F(x, a) = \lim_{\kappa\to\infty} F^{[\kappa]}(x, a)$. We prove Theorem 5.2 in the case $\kappa = 2$ in the appendix by exploiting the relative simplicity of $F^{[2]}(x, a)$ to prove (60) uniformly in ℓ.

6. An analogue to Ramanujan's theorem

In this section, we prove an ineffective and then an effective version of Ramanujan's theorem for $\sigma^{[\kappa]}$. The following material is inspired by [3, Chapter 7].

6.1. An ineffective theorem.

Our goal in this subsection is to prove the following analogue to Ramanujan's theorem, which gives the implication (1) $\Longrightarrow$ (2) in Theorem 1.5.

Theorem 6.1. *Let $\kappa > 3/2$ be a real number. If the Riemann hypothesis holds, then*

$$\sigma^{[\kappa]}(n) < \frac{(e^\gamma n \log\log n)^\kappa}{\zeta(\kappa)} \tag{61}$$

for all n sufficiently large.

We first prove that we can restrict our attention to κ-colossally abundant numbers.

Lemma 6.2. *Let i_0 be a positive integer. If* (61) *holds for $n = N_i^{[\kappa]}$ for all $i \geq i_0$, then* (61) *holds for all $n \geq N_{i_0}^{[\kappa]}$.*

Proof. Let $n \geq N^{[\kappa]}(i_0)$. By Corollary 4.27, there are κ-colossally abundant numbers $N_i^{[\kappa]}$ and $N_{i+1}^{[\kappa]}$ with

$$N_{i_0}^{[\kappa]} \leq N_i^{[\kappa]} \leq n \leq N_{i+1}^{[\kappa]}.$$

Suppose n violates (61). In the language of Definition 4.28, this means $1 \leq G^{[\kappa]}(n)$. Then by Theorem 4.29, we see

$$1 \leq G^{[\kappa]}(n) \leq \max\left(G^{[\kappa]}\left(N_i^{[\kappa]}\right), G^{[\kappa]}\left(N_{i+1}^{[\kappa]}\right)\right),$$

so at least one of $N_i^{[\kappa]}$ and $N_{i+1}^{[\kappa]}$ also violates Robin's inequality. Iterating this argument with n larger than $N_{i+2}^{[\kappa]}$, our claim follows. ☐

We are now in a position to prove Theorem 6.1.

Proof of Theorem 6.1. If there is some $n = N$ violating (61), then by Lemma 6.2, we may assume $N = N^{[\kappa]}(\epsilon)$ for some ϵ. Writing x_ℓ for $x_\ell^{[\kappa]}(\epsilon)$, (56) tells us

$$\frac{\sigma^{[\kappa]}(N)}{N^\kappa} = \prod_{\ell\geq 1} \prod_{x_{\ell+1}<p\leq x_\ell} \frac{\sigma^{[\kappa]}(p^\ell)}{p^{\kappa\ell}}.$$

Splitting off the first factor from the rest, and applying Lemma 3.10, we have

$$\frac{\sigma^{[\kappa]}(N)}{N^\kappa} < \prod_{x_2<p\leq x} \frac{\sigma^{[\kappa]}(p)}{p^\kappa} \prod_{p\leq x_2} \frac{(1-p^{-\kappa})}{(1-p^{-1})^\kappa}.$$

Let $x \coloneqq x_1$, and assume $x \geq 2^{15}$. Expanding $\sigma^{[\kappa]}(p)/p^\kappa$ and rearranging, we obtain

$$\frac{\sigma^{[\kappa]}(N)}{N^\kappa} < \prod_{p \leq x} \left(1 - p^{-1}\right)^{-\kappa} \prod_{x_2 < p \leq x} \left(1 - p^{-2}\right)^\kappa \prod_{p \leq x} \left(1 - p^{-\kappa}\right) \prod_{x_2 < p \leq x} \frac{\left(1 - (p+1)^{-\kappa}\right)}{1 - p^{-\kappa}}. \tag{62}$$

Let us consider each of the above products in turn. By Lemma 2.9 and (57), we have

$$\prod_{p \leq x} \left(1 - p^{-1}\right)^{-1} \leq e^\gamma \log\log N \exp\left(\frac{2 + \beta - 0.977\sqrt{2}}{\sqrt{x}\log x} + O\left(\frac{\alpha(x)}{\sqrt{x}\,(\log x)^2}\right)\right), \tag{63}$$

where $\beta = \gamma + 2 - \log 4\pi = 0.04619...$ and $\alpha(x)$ is as in (10).

By Lemma 4.13 and Lemma 2.6, we see

$$\prod_{x_2 < p \leq x} \left(1 - p^{-2}\right) \leq \prod_{\sqrt{2x} < p \leq x} \left(1 - p^{-2}\right) \leq \exp\left(-\frac{\sqrt{2}}{\sqrt{x}\log x} + \frac{4}{\sqrt{x}\,(\log x)^2}\right). \tag{64}$$

By Lemma 2.3, we get

$$\prod_{p \leq x} \left(1 - p^{-\kappa}\right) < \frac{1}{\zeta(\kappa)} \exp\left(\frac{1.01624\kappa x^{1-\kappa}}{\log x}\left(\frac{1}{\kappa - 1} + 0.000052\right)\right). \tag{65}$$

Finally, since $\sqrt{x} < x_2$ according to Lemma 4.13, we find by Lemma 2.5 that

$$\prod_{x_2 < p \leq x} \frac{1 - (p+1)^{-\kappa}}{1 - p^{-\kappa}} < \prod_{p > \sqrt{x}} \frac{1 - (p+1)^{-\kappa}}{1 - p^{-\kappa}} < \exp\left(\frac{2.1558(\kappa + 1)}{x^{\kappa/2}\log x}\right). \tag{66}$$

Substituting (63), (64), (65), and (66) into (62), we obtain

$$\begin{aligned}\frac{\sigma^{[\kappa]}(N)}{N^\kappa(\log\log N)^\kappa} &< \frac{e^{\kappa\gamma}}{\zeta(\kappa)} \exp\left(\frac{(2 + \beta - 1.977\sqrt{2})\kappa}{\sqrt{x}\log x} + O\left(\frac{1}{\sqrt{x}\,(\log x)^2}\right)\right) \\ &< \frac{e^{\kappa\gamma}}{\zeta(\kappa)} \exp\left(\frac{-0.7497\kappa}{\sqrt{x}\log x} + O\left(\frac{1}{\sqrt{x}\,(\log x)^2}\right)\right).\end{aligned}$$

For x sufficiently large, this expression is always less than $e^{\kappa\gamma}/\zeta(\kappa)$. ☐

6.2. **An effective theorem.**

Although Theorem 6.1 is quite general, it lacks the effectiveness asserted in Theorem 1.5. We adapt the proof of Theorem 6.1 and leverage Lemma 4.17 to prove the following special case of Theorem 1.5. Combined with Lemma 5.3, this will yield our main theorem.

Theorem 6.3. *If the Riemann hypothesis holds, then*

$$\sigma^{[2]}(n) < \frac{(e^\gamma n \log\log n)^2}{\zeta(2)} \tag{67}$$

for all $n > 2162160$.

Remark 6.4. There are a total of 79 counterexamples to (67) among $1 < n < 10^{10^{12.1408}}$. We list them now: 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 15, 16, 18, 20, 24, 28, 30, 36, 40, 42, 48, 54, 60, 72, 84, 90, 96, 108, 120, 126, 132, 144, 168, 180, 210, 240, 252, 300, 336, 360, 420, 480, 504, 540, 600, 630, 660, 720, 840, 1080, 1260, 1320, 1440, 1680, 2520, 3360, 3780, 4200, 4620, 5040, 7560, 9240, 10080, 12600, 13860, 15120, 18480, 27720, 32760, 55440, 65520, 83160, 110880, 166320, 360360, 720720, 1441440, 2162160.

To verify that no counterexamples exist past 2162160 up to $10^{10^{12.1408}}$, we used the code generously supplied by Platt to calculate the result of [14, Theorem 5], replacing the functions related to σ with the analogous ones for $\sigma^{[2]}$. The number $10^{10^{12.1408}}$ is the approximate value of the largest 2-CA number we have checked which satisfies inequality (67) in the 2-Robin criterion. Specifically, this 2-CA number has prime factorization

$$2^{45}3^{28}5^{19}7^{16}11^{12}13^{12}17^{10}19^{10}23^{9}29^{9}[31\cdots43]^{8}[47\cdots79]^{7}[83\cdots157]^{6}[163\cdots431]^{5}$$
$$[433\cdots1867]^{4}[1871\cdots20963]^{3}[20981\cdots2495887]^{2}[2495947\cdots3187251775319].$$

Here we use $[p\cdots q]$ to denote the product of consecutive primes from p to q, inclusive.

Proof of Theorem 6.3. Let $\kappa = 2$. We follow the proof of Theorem 6.1 exactly, save that we replace the ineffective inequality (63) with the effective inequality

$$\prod_{p\le x}\left(1-p^{-1}\right)^{-1} < e^{\gamma}\log\log N \exp\left(\frac{2+\beta-0.977\sqrt{2}}{\sqrt{x}\log x}+\frac{2\alpha(x)+0.977\sqrt{2}\log 2}{2\sqrt{x}\,(\log x)^{2}}\right), \tag{68}$$

using Lemma 2.9 and (58). Substituting (64), (65), (66), and (68) into (62), we obtain an expression of the form

$$\frac{\sigma^{[\kappa]}(N)}{N^{\kappa}(\log\log N)^{\kappa}} < \frac{e^{\kappa\gamma}}{\zeta(\kappa)}\exp\left(\frac{-1.4994+\epsilon(x)}{\sqrt{x}\log x}\right),$$

where

$$\epsilon(x) := \frac{2\alpha(x)+0.977\sqrt{2}\log 2+4}{\log x}+\frac{8.5}{\sqrt{x}}$$

is decreasing in x on $[2^{15},\infty)$. But $\epsilon(2^{15}) < 1.4994$ by Lemma 2.7, so (67) holds when $x \ge 2^{15}$. A short computation finishes the proof. ☐

Corollary 6.5. *Let $\kappa \ge 2$. If the Riemann hypothesis holds, then*

$$\sigma^{[\kappa]}(n) < \frac{(e^{\gamma}n\log\log n)^{\kappa}}{\zeta(\kappa)} \tag{69}$$

for all $n > 2162160$.

Proof. Let $\kappa \ge 2$. By Lemma 5.3, we have

$$\left(\zeta(\kappa)\sigma^{[\kappa]}(n)\right)^{1/\kappa} \le \left(\zeta(2)\sigma^{[2]}(n)\right)^{1/2}.$$

But by Theorem 6.3,

$$\left(\zeta(2)\sigma^{[\kappa]}(n)\right)^{1/2} < e^{\gamma}n\log\log n$$

whenever $n > 2162160$. The claim follows. ☐

When $\kappa \ge 2$, Corollary 6.5 furnishes the implication (1) $\Longrightarrow$ (2′) in Theorem 1.5, concluding its proof.

7. An analogue to Lagarias's criterion

Lagarias proved that the Riemann hypothesis is equivalent to the claim that the inequality (8) holds for $n > 1$ [8, Theorem 1.1]. He verified his criterion by means of the following lemma.

Lemma 7.1 (Lagarias' Lemma)**.** *For $n \ge 20$, we have*

$$e^{\gamma}n\log\log n + H_n \le H_n + e^{H_n}\log H_n \le e^{\gamma}n\log\log n + \frac{7n}{\log n},$$

where $H_n := \sum_{1\le m\le n} 1/m$ is the nth harmonic number.

Proof. [8, Lemmas 3.1 and 3.2] (see also [3, Lemma 7.17]). ☐

We use Lemma 7.1 to formulate an analogue to Lagarias' criterion.

Theorem 7.2. *Let $\kappa \geq 2$ be given. The following are equivalent:*

1. *The Riemann hypothesis holds;*
2. *For $n > 55440$, we have*

$$\sigma^{[\kappa]}(n) < \frac{\left(e^{H_n} \log H_n\right)^{\kappa}}{\zeta(\kappa)}; \tag{70}$$

3. *For $n > 55440$, we have*

$$\sigma^{[\kappa]}(n) < \frac{\left(H_n + e^{H_n} \log H_n\right)^{\kappa}}{\zeta(\kappa)}. \tag{71}$$

Proof. (1) $\Longrightarrow$ (2). By Theorem 6.3 and Lemma 7.1, if the Riemann hypothesis holds, we have

$$\sigma^{[\kappa]}(n) < \frac{(e^{\gamma} n \log\log n)^{\kappa}}{\zeta(\kappa)} \leq \frac{\left(e^{H_n} \log H_n\right)^{\kappa}}{\zeta(\kappa)}. \tag{72}$$

which proves the first implication for $n > 2162160$. The implication follows from a direct comparison between the first two terms in (72) for $1 \leq n \leq 2160160$.

(2) $\Longrightarrow$ (3). This is immediate.

(3) $\Longrightarrow$ (1). Suppose by contradiction that for all n sufficiently large, we have

$$\sigma^{[\kappa]}(n) < \frac{\left(H_n + e^{H_n} \log H_n\right)^{\kappa}}{\zeta(\kappa)},$$

but the Riemann hypothesis fails. By Theorem 5.2 and the inequality $e^{x/2} < 1 + x$ for all $x \in (0, 2)$, there is a real number b with $0 < b < \frac{1}{2}$ and a constant $c > 0$ such that

$$\frac{(e^{\gamma} n \log\log n)^{\kappa}}{\zeta(\kappa)} \exp\left(\frac{c}{(\log n)^{b}}\right) < \sigma^{[\kappa]}(n)$$

for infinitely many n. On the other hand, by Lemma 7.1 and by assumption, the following inequalities hold for all n sufficiently large:

$$\begin{aligned}\sigma^{[\kappa]}(n) &< \frac{\left(H_n + e^{H_n} \log H_n\right)^{\kappa}}{\zeta(\kappa)} \\ &< \frac{(e^{\gamma} n \log\log n)^{\kappa}}{\zeta(\kappa)} \left(1 + \frac{7}{e^{\gamma} \log n \log\log n}\right)^{\kappa} \\ &< \frac{(e^{\gamma} n \log\log n)^{\kappa}}{\zeta(\kappa)} \exp\left(\frac{7\kappa}{e^{\gamma} \log n \log\log n}\right).\end{aligned}$$

But for all n sufficiently large, we have

$$\frac{c}{(\log n)^{b}} > \frac{7\kappa}{e^{\gamma} \log n \log\log n},$$

so this is a contradiction. □

Remark 7.3. When $\kappa = 2$, there are a total of 33 counterexamples to (70) among $1 < n < 10^{10^{12.1408}}$. We list them now: 2, 3, 4, 6, 8, 10, 12, 18, 20, 24, 30, 36, 42, 48, 60, 72, 84, 90, 120, 168, 180, 240, 360, 420, 720, 840, 1260, 1680, 2520, 5040, 27720, 55440.

Similarly, when $\kappa = 2$, there are a total of 25 counterexamples to (71) among $1 < n < 10^{10^{12.1408}}$. We list them now: 2, 4, 6, 12, 18, 24, 30, 36, 48, 60, 72, 84, 120, 180, 240, 360, 420, 720, 840, 1260, 1680, 2520, 5040, 27720, 55440.

Theorem 7.2 demands that $n > 55440$, but we need enlarge κ only a little bit before the counterexamples we have enumerated disappear.

Corollary 7.4. *Given $\kappa \geq 3.89$, the following are equivalent:*

- *The Riemann hypothesis holds;*
- *For $n > 1$, we have*

$$\sigma^{[\kappa]}(n) < \frac{\left(H_n + e^{H_n}\log H_n\right)^{\kappa}}{\zeta(\kappa)}. \tag{73}$$

Proof. When $\kappa = 3.89$, direct computation shows that (73) holds for each of the counterexamples listed in the remark above. By Lemma 5.3, (73) also holds for every $\kappa \geq 3.89$. The corollary now follows from Theorem 7.2. □

Of course, Theorem 6.1 also furnishes us with an ineffective version of the Lagarias criterion for $\kappa > 3/2$.

Theorem 7.5. *Let $\kappa > 3/2$ be given. The following are equivalent:*

1. *The Riemann hypothesis holds;*
2. *For n sufficiently large, we have*
$$\sigma^{[\kappa]}(n) < \frac{\left(e^{H_n}\log H_n\right)^{\kappa}}{\zeta(\kappa)};$$
3. *For n sufficiently large, we have*
$$\sigma^{[\kappa]}(n) < \frac{\left(H_n + e^{H_n}\log H_n\right)^{\kappa}}{\zeta(\kappa)}.$$

8. Future work

We believe that our work makes a promising start on the study of $\sigma^{[\kappa]}(n)$, but much work remains to be done. A natural question is: can we reduce the threshold for κ in Corollary 6.5 to 3/2?

We also wish to highlight the discrepancy between the ranges of κ in Theorem 6.1 and Theorem 5.2. Our proof of Theorem 6.1 fails for $1 < \kappa \leq 3/2$ because the right-hand side of (65) dominates the other terms in (62). Nonetheless, it would be interesting to know whether Theorem 6.1 still holds for smaller values of κ.

All the work in this paper presumes that $\kappa > 1$, but it would be both natural and interesting to study $\sigma^{[\kappa]}(n)$ for $\kappa \leq 1$. By (19), we see

$$\sigma^{[\kappa]}(p^{\ell})\sigma^{[-\kappa]}(p^{\ell}) = \left(1 - u^{[\kappa]}(p,\ell)\right)\left(1 - u^{[-\kappa]}(p,\ell)\right),$$

where

$$u^{[\kappa]}(p,\ell) := \left(\frac{1-p^{-\ell}}{p-p^{-\ell}}\right)^{\kappa} = \frac{1}{u^{[-\kappa]}(p,\ell)}.$$

Thus, we expect some sort of duality between the behavior of $\sigma^{[\kappa]}(n)$ and $\sigma^{[-\kappa]}(n)$. The details of this correspondence, however, require further study.

Our analysis of $\sigma^{[\kappa]}(n)$ depended on our study of κ-colossally abundant numbers in Section 4. We noted after Definition 4.20 that if

$$E_p^{[\kappa]} \cap E_q^{[\kappa]} \neq \emptyset \tag{74}$$

then

$$\left\{\, N^{[\kappa]}(\epsilon) \,\right\}_{\epsilon>0} \subsetneq \left\{\, N \in \mathbb{Z}_{>0} \;:\; N \text{ is } \kappa\text{-colossally abundant} \,\right\}$$

(see Example 4.21). For $p \neq q$ fixed primes, can we find κ for which (74) holds? Does (74) hold for any large κ? Does (74) hold for any integral κ?

The inequality (6) is equivalent to the statement

$$\sigma_{-1}(n) < e^{\gamma} \log\log n,$$

which suggests an entirely different line of inquiry: rather than studying $\sigma^{[\kappa]}(n)$, we could study

$$\sigma_{-1}^{[\kappa]}(n) = \sum_{d|n} \mu(n/d)\sigma_{-1}^{\kappa}(d),$$

which satisfies

$$\sigma_{-1}^{[k]}(n) = \sum_{[d_1,\dots,d_k]=n} \frac{1}{d_1 \dots d_k}$$

whenever $\kappa = k \geq 1$ is an integer. In (17) we determined an asymptotic for the partial sums of $\sigma_{-1}^{[2]}(n)$; we wish to generalize this asymptotic to all $\kappa > 1$. It is not hard to show that for any $\kappa > 1$ we have

$$\limsup_{n\to\infty} \frac{n\sigma_{-1}^{[\kappa]}(n)}{\kappa^{\omega(n)}(e^{\gamma}\log\log n)^{\kappa-1}} = 1.$$

This analogue to (5) and Theorem 1.4 invites investigation, and we are interested in developing an analogue to the Ramanujan–Robin criterion for $\sigma_{-1}^{[\kappa]}(n)$.

More generally, one can investigate κ-analogues of Robin's criterion (6) for other arithmetic functions of interest. For example, Nicolas [15, 16] showed that the Riemann hypothesis implies that $\varphi(n) < e^{-\gamma} n/\log\log n$ for all primorials n and that the falsity of the Riemann hypothesis would imply the reversed inequality for infinitely many primorials n. It would be interesting to prove analogues of Theorem 1.5 for

$$f^{[\kappa]}(n) := \sum_{d|n} \mu\left(\frac{n}{d}\right) f(d)^{\kappa}$$

with $f(d) = \varphi(d)$, and perhaps also with its cousin $f(d) = \psi(d)$, where

$$\psi(d) := n \prod_{p|d} \left(1 + \frac{1}{p}\right)$$

is the Dedekind totient function.

In addition, despite the fact that Robin's inequality (6) is out of reach, Luca, Pomerance and Solé [12] have considered the exceptional set of positive integers n for (6). They showed that the number of $n \leq x$ for which (6) fails is at most $x^{O(1/\log\log x)}$ for $x \geq 3$. It may be of interest to consider κ-analogues of their results for (7).

Finally, Washington and Yang [26] and Vega [24] have published variants of the Ramanujan–Robin criterion where the domain of $\sigma(n)$ is restricted to special prime factorizations. Applying their methods to our functions furnishes another natural line of inquiry.

APPENDIX: ROBIN'S THEOREM WHEN $\kappa = 2$

Robin's proof [19] of Theorem 5.1 could be adapted to prove Theorem 5.2 without recourse to Lemma 5.3, if we could somehow prove the inequality

$$x_a > x^{1/a} \tag{75}$$

for $a \geq 2$.

Indeed, for colossally abundant N, Robin [19, page 205] writes

$$\frac{\sigma(N)}{N} = \prod_{p\le x}\left(1-p^{-1}\right)^{-1}\prod_{x_2<p\le x}\left(1-p^{-2}\right)\prod_{\ell\ge 2}\prod_{x_{\ell+1}<p\le x_\ell}\left(1-p^{-\ell-1}\right). \tag{76}$$

Similarly, for κ-colossally abundant N, we have

$$\frac{\sigma^{[\kappa]}(N)}{N^\kappa} = \frac{1}{\zeta(\kappa)}\left(\prod_{p\le x}\left(1-p^{-1}\right)^{-1}\prod_{x_2<p\le x}\left(1-p^{-2}\right)\prod_{\ell\ge 2}\prod_{x_{\ell+1}<p\le x_\ell}\left(1-p^{-\ell-1}\right)\right)^{\kappa}E_\kappa(x), \tag{77}$$

where

$$E_\kappa(x) \coloneqq \zeta(\kappa)\prod_{\ell\ge 1}\prod_{x_{\ell+1}<p\le x_\ell}\left(1-\left(\frac{1-p^{-\ell}}{p-p^{-\ell}}\right)^{\kappa}\right). \tag{78}$$

For $\kappa \ge 2$, a straightforward application of the prime number theorem to (78) shows unconditionally that

$$E_\kappa(x) = \exp\left(O\left(\frac{1}{x\log x}\right)\right)$$

for x sufficiently large, so the contribution of $E_\kappa(x)$ to (77) is negligible. The remaining terms in (77) are in visible correspondence with the terms in (76). Robin's handling of $\prod_{p\le x}\left(1-p^{-1}\right)^{-1}$ goes through exactly, and by Lemma 4.15 his handling of $\prod_{x_2<p\le x}\left(1-p^{-2}\right)$ does as well. However, without (75), we cannot follow Robin in handling

$$\prod_{\ell\ge 2}\prod_{x_{\ell+1}<p\le x_\ell}\left(1-p^{-\ell-1}\right).$$

In the remainder of this appendix, we prove (75) holds when $\kappa = 2$. By the argument sketched above, this gives us a direct proof of the 2-analogue of Robin's theorem.

By the monotonicity of $F_\kappa(x,a)$ in x, (75) is equivalent to showing

$$F_\kappa(x_a,a) \le F_\kappa(x^{1/a},a),$$

which is equivalent to

$$F_\kappa(x_1,1) \le F_\kappa(x^{1/a},a).$$

This simplifies to

$$(x+1)^\kappa - 1 \le \left[\frac{(x^{1+1/a}-1)^\kappa-(x-1)^\kappa}{(x-1)^\kappa-(x^{1-1/a}-1)^\kappa}\right]^a. \tag{79}$$

Specializing now to the case $\kappa = 2$, (79) becomes

$$(x+1)^2 - 1 \le \left[\frac{(x^{1+1/a}-1)^2-(x-1)^2}{(x-1)^2-(x^{1-1/a}-1)^2}\right]^a.$$

Letting $y^a = x$, we have

$$(y^a+1)^2 - 1 \le \left[\frac{(y^{a+1}-1)^2-(y^a-1)^2}{(y^a-1)^2-(y^{a-1}-1)^2}\right]^a.$$

We then factor each of the differences of two squares:

$$y^a(y^a+2) \le \left[\frac{(y^{a+1}+y^a-2)(y^{a+1}-y^a)}{(y^a+y^{a-1}-2)(y^a-y^{a-1})}\right]^a = y^a\left(\frac{y^{a+1}+y^a-2}{y^a+y^{a-1}-2}\right)^a.$$

Thus it remains to show that

$$y^a+2 \le \left(\frac{y^{a+1}+y^a-2}{y^a+y^{a-1}-2}\right)^a$$

for $y^a \geq 2$. This inequality rearranges to

$$p(y) \coloneqq (y^{a+1} + y^a - 2)^a - (y^a + 2)(y^a + y^{a-1} - 2)^a \geq 0,$$

so we show that the polynomial $p(y) \geq 0$ for $y^a \geq 2$. When $a = 2$, one may verify directly that $p(y) = 2(y^2 - 2)(y - 1)^2 \geq 0$ for $y^2 \geq 2$. For $a > 2$, observe that this polynomial has root 1 with multiplicity a and root $2^{1/a}$ with multiplicity at least 1. We will show that in fact these are all of the positive roots by using Descartes' rule of signs and showing that the coefficients of this polynomial has exactly $a + 1$ sign changes. Then it follows that $p(y) \geq 0$ for $y^a \geq 2$.

Using the binomial theorem, we find that the leading term of $p(y)$ is $2(a - 1)y^{a^2}$. Writing the ith coefficient of $p(y)$ as c_i, we thus have $c_{a^2} = 2(a - 1)$, and that $c_0 = p(0) = -(-2)^a$.

For the remaining terms, we use the binomial expansions

$$(y^{a+1} + y^a - 2)^a = \sum_{i=0}^{a} \sum_{j=0}^{i} \binom{a}{i}\binom{i}{j}(-2)^{a-i} y^{ai+j},$$

$$-y^a(y^a + y^{a-1} - 2)^a = -\sum_{i=0}^{a} \sum_{j=0}^{i} \binom{a}{i}\binom{i}{j}(-2)^{a-i} y^{ai+a-i+j},$$

$$-2(y^a + y^{a-1} - 2)^a = \sum_{i=0}^{a} \sum_{j=0}^{i} \binom{a}{i}\binom{i}{j}(-2)^{a-i+1} y^{ai-i+j}.$$

By matching up the powers of y, we can determine the coefficients c_m of p. We find for $0 \leq m < a - 1$ that

$$c_{am} = (-2)^{a-m}\left(2\binom{a}{m-1} - \binom{a}{m}\right),$$

along with the special case $m = a - 1$ where $c_{a(a-1)} = -2(a - 1)^2$. When $1 \leq r < a$, we have

$$c_{am+r} = (-2)^{a-m}\left[\binom{a}{m}\binom{m}{r} - \binom{a}{m}\binom{m}{r-a+m} + \binom{a}{m+1}\binom{m+1}{r-a+m+1}\right].$$

We may simplify the formula for c_{am} when $m \neq 0$ by writing

$$\binom{a}{m} = \frac{a-m+1}{m}\binom{a}{m-1},$$

so

$$c_{am} = (-2)^{a-m}\left(2 - \frac{a-m+1}{m}\right)\binom{a}{m-1} = (-2)^{a-m}\frac{3m-a-1}{m}\binom{a}{m-1}.$$

Thus, the sign of c_{am} follows the sign of $(-1)^{a-m}(3m - a - 1)$.

We now show that for each m satisfying $0 \leq m \leq a - 1$, as r increases from 1 to $a - 1$ the bracketed factor of c_{am+r} always starts positive (or zero) and switches to negative at most once.

But as

$$\binom{a}{m+1} = \frac{a-m}{m+1}\binom{a}{m},$$

we observe that the sign of the bracket follows the sign of

$$B = B(r, m, a) = \binom{m}{r} - \binom{m}{r-a+m} + \frac{a-m}{m+1}\binom{m+1}{r-a+m+1},$$

so that

$$c_{am+r} = (-2)^{a-m}\binom{a}{m}B.$$

First observe that since $a \geq m$, B can only be negative if the central term is nonzero. This means $a - m \leq r \leq a$. We now focus on the difference

$$D = -\binom{m}{r-a+m} + \frac{a-m}{m+1}\binom{m+1}{r-a+m+1}.$$

Since both binomial coefficients are nonzero when $a - m \leq r \leq a$, we have

$$D = \left(\frac{a-m}{r-a+m+1} - 1\right)\binom{m}{r-a+m} = \frac{2a-2m-r-1}{r-a+m+1}\binom{m}{r-a+m}.$$

The sign of D follows the sign of the fraction. Since in our case the denominator is positive, D is negative precisely when $r > 2a - 2m - 1$.

Finally, we determine when B is negative by checking when we have

$$\binom{m}{r} < \frac{r-2a+2m+1}{r-a+m+1}\binom{m}{r-a+m}.$$

Since the right side is positive, the inequality certainly holds when the left side is zero, that is, when $r > m$. Now suppose the left side is positive, so $r \leq m$. Then we have

$$\frac{m!}{r!(m-r)!} < \frac{r-2a+2m+1}{r-a+m+1}\frac{m!}{(r-a+m)!(a-r)!},$$

$$\frac{(a-r)!}{(m-r)!} < \frac{r!(r-2a+2m+1)}{(r-a+m+1)!},$$

from which it follows that

$$(a-r)(a-r-1)\cdots(m-r+1) < r(r-1)\cdots(r-a+m+2)(r-2a+2m+1).$$

Observe that each factor in the above inequality is nonnegative, and as r increases, the left side decreases and the right side increases. Thus the direction of the inequality switches at most once, depending on the direction of the inequality when $r = m$. Therefore, the sign of B changes at most once, and this change can be detected when B is negative for $r = a - 1$. Explicitly, we find that

$$B = B(a-1, m-1, a) = -\binom{m-1}{m-2} + \frac{a-m+1}{m}\binom{m}{m-1} = a - 2m + 2 < 0$$

when

$$m > \frac{a+2}{2}.$$

To summarize, we have shown that for fixed m satisfying $0 \leq m < a$, as r increases from 1 to $a - 1$, the bracketed factor of c_{am+r} starts positive (or zero for $m = 0$) and switches to negative at most once.

It remains to compare the signs of c_{am-1}, c_{am}, and c_{am+1} for each m between 1 and $a - 1$. We have that c_{am} has the same sign as $(-1)^{a-m}$ if $m > (a+1)/3$, is zero if $m = (a+1)/3$, and has the opposite sign if $m < (a+1)/3$. For $r = 1$ and $m = 0$ we have $c_1 = 0$, and for $m > 0$ the sign of c_{am+1} is $(-1)^{a-m}$. Finally for c_{am-1}, we have that the sign is $(-1)^{a-m}$ for $a/2 < m - 1$, zero for $a/2 = m - 1$, and opposite for $a/2 > m - 1$.

We can now tally the sign changes of c_i. Observe that there is a one-to-one correspondence between coefficients $c_{am}, 0 \leq m \leq a$ and the sign changes. It may help to refer to the figure below showing the case $a = 20$ to see this.

In this figure we arrange the coefficients of $p(y)$ in a square array, starting with c_0 in the upper left corner, and then c_r with r increasing to the right from $r = 1$ to $r = a - 1 = 19$. The next row represents coefficients c_a to c_{2a-1}, and so on. There is one extra coefficient that lives below the lower left corner for c_{a^2}. The sign of B for each coefficient is labeled,

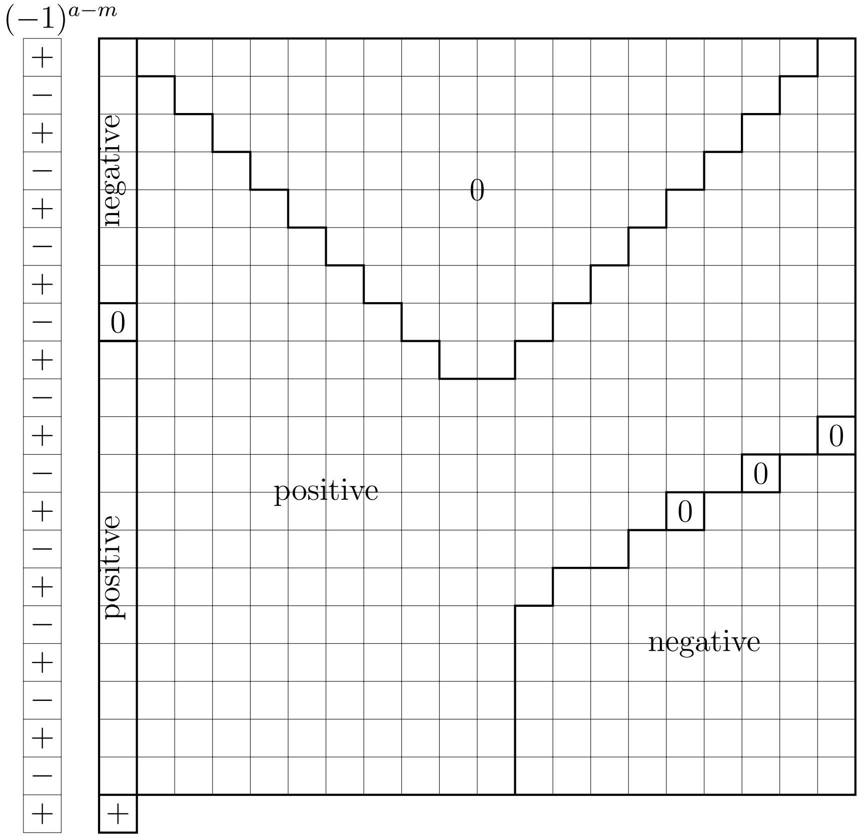

FIGURE 1. The signs of the coefficients of $p(y)$ in the case $a = 20$.

and the signs of $(-1)^{a-m}$ are in a column on the left side for reference. Starting from the the leftmost column, we initially have sign changes when we leave the column. Then starting at the zero which occurs when $m = a/2 = 10$, the sign changes as we cross or enter the column. Then when $m - 1 \geq (a+1)/3 = 7$, or $m = 8$, the behavior changes again. Now the sign changes start occurring earlier, but can still be made to pair up with entries in the leftmost column. Therefore, there are $a + 1$ sign changes, establishing the original claim.

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