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Numerically explicit estimates for the distribution of rough numbers

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A B S T R A C T

For $x \geq y > 1$ and $u := \log x/\log y$, let $\Phi(x, y)$ denote the number of positive integers up to x free of prime divisors less than or equal to y. In 1950 de Bruijn [4] studied the approximation of $\Phi(x, y)$ by the quantity

$$\mu_y(u)e^{\gamma}x\log y\prod_{p\leq y}\left(1-\frac{1}{p}\right),$$

where $\gamma = 0.5772156...$ is Euler's constant and

$$\mu_y(u) := \int_1^u y^{t-u}\omega(t)\,dt.$$

He showed that the asymptotic formula

$$\Phi(x,y) = \mu_y(u)e^{\gamma}x\log y\prod_{p\leq y}\left(1-\frac{1}{p}\right) + O\left(\frac{xR(y)}{\log y}\right)$$

holds uniformly for all $x \geq y \geq 2$, where $R(y)$ is a positive decreasing function related to the error estimates in the Prime

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Number Theorem. In this paper we obtain numerically explicit versions of de Bruijn's result.

1. Introduction

Let $x \geq y > 1$ be positive real numbers. Throughout the paper, we shall always write $u := \log x/\log y$, and the letters p and q will always denote primes. We say that a positive integer n is *y-rough* if all the prime divisors of n are greater than y. Let $\Phi(x, y)$ denote the number of y-rough numbers up to x. Explicitly, we have

$$\Phi(x, y) = \sum_{\substack{n \leq x \\ P^-(n) > y}} 1,$$

where $P^-(n)$ denotes the least prime divisor of n, with the convention that $P^-(1) = \infty$. When $1 \leq u \leq 2$, or equivalently when $\sqrt{x} \leq y \leq x$, we simply have $\Phi(x, y) = \pi(x) - \pi(y) + 1$, where $\pi(\cdot)$ is the prime-counting function. The function $\Phi(x, y)$ is closely related to the sieve of Eratosthenes, one of the most ancient algorithms for finding primes, and it has been extensively studied by mathematicians. A simple application of the inclusion-exclusion principle enables us to write

$$\Phi(x, y) = \sum_{d \mid P(y)} \mu(d) \left\lfloor \frac{x}{d} \right\rfloor, \tag{1.1}$$

where $\lfloor a \rfloor$ is the integer part of a for any $a \in \mathbb{R}$, μ is the Möbius function, and $P(y)$ denotes the product of primes up to y. If y is relatively small in comparison with x, say $y = x^{o(1)}$, the above formula can be used to obtain $\Phi(x, y) \sim e^{-\gamma} x/\log y$ as $y \to \infty$, where $\gamma = 0.5772156...$ is Euler's constant. However, it turns out that this nice asymptotic formula does not hold uniformly, as already exemplified by the base case $1 \leq u \leq 2$.

In 1937, Buchstab [3] showed that for any fixed $u > 1$, one has $\Phi(x, y) \sim \omega(u)x/\log y$ as $x \to \infty$, where $\omega(u)$ is defined to be the unique continuous solution to the delay differential equation $(u\omega(u))' = \omega(u-1)$ for $u \geq 2$, subject to the initial value condition $\omega(u) = 1/u$ for $u \in [1, 2]$. Comparing this result with the asymptotic formula obtained from (1.1), one would expect that $\omega(u) \to e^{-\gamma}$ as $u \to \infty$. Indeed, it can be shown [13, Corollary III.6.5] that $\omega(u) = e^{-\gamma} + O(u^{-u/2})$ for $u \geq 1$. Moreover, it is known that $\omega(u)$ oscillates above and below $e^{-\gamma}$ infinitely often. It is convenient to extend the definition of $\omega(u)$ by setting $\omega(u) = 0$ for all $u < 1$, so that $\omega(u)$ satisfies the same delay differential equation on $\mathbb{R} \setminus \{1, 2\}$. In the sequel, we shall write $\omega'(1)$ and $\omega'(2)$ for the right derivatives of $\omega(u)$ at $u = 1$ and $u = 2$, respectively. With this convention, we have $(u\omega(u))' = \omega(u-1)$ for all $u \in \mathbb{R}$.

Buchstab's asymptotic formula can be proved easily based on the following identity [13, Theorem III.6.3] named after him:

$$\Phi(x, y) = \Phi(x, z) + \sum_{y<p\le z} \sum_{v\ge 1} \Phi(x/p^v, p) \tag{1.2}$$

for any $z \in [y, x]$. The Buchstab function $\omega(u)$ then appears naturally in the iteration process, starting with $\Phi(x, y) \sim x/(u \log y)$ in the range $1 < u \le 2$. Since $1/2 \le \omega(u) \le 1$ for $u \in [1, \infty)$, Buchstab's asymptotic formula suggests that the relation $\Phi(x, y) \asymp x/\log y$ holds uniformly for $x \ge y > 1$. Thus, it is of interest to seek numerically explicit estimates for $\Phi(x, y)$ that are applicable in wide ranges. Confirming a conjecture of Ford, the author [6] showed that $\Phi(x, y) < x/\log y$ holds uniformly for $x \ge y > 1$, which is essentially best possible when $x^{1-\epsilon} \le y \le \epsilon x$, where $\epsilon \in (0, 1)$ is fixed. On the other hand, the values of $\omega(u)$ indicate that improvements should be expected in the narrower range $2 \le y \le \sqrt{x}$. In recent work jointly with Pomerance [7], the author proved that $\Phi(x, y) < 0.6x/\log y$ holds uniformly for $3 \le y \le \sqrt{x}$. This inequality provides a fairly good upper bound for $\Phi(x, y)$, especially considering that the absolute maximum of $\omega(u)$ over $[2, \infty)$ is given by $M_0 = 0.5671432...$, attained at the unique critical point $u = 2.7632228...$ of the function $(\log(u-1)+1)u^{-1}$ on $[2, 3]$. With a bit more effort, one can show, using the Buchstab identity (1.2), that

$$\Phi(x, y) = \frac{x}{\log y}\left(\omega(u) + O\left(\frac{1}{\log y}\right)\right) \tag{1.3}$$

uniformly for $2 \le y \le \sqrt{x}$ (see [13, Theorem III.6.4]). In Section 2, we shall derive a numerically explicit lower bound of this type that suits our needs. Our method can also be modified with ease to obtain a numerically explicit upper bound of the same type.

In [4] de Bruijn provided a more precise approximation for $\Phi(x, y)$ than $\omega(u)x/\log y$. Let us fix some $y_0 \ge 2$ for the moment. Suppose that there exist a positive constant $C_0(y_0)$ and a positive decreasing function $R(z)$ defined on $[y_0, \infty)$, such that $R(z) \gg z^{-1}$, that $R(z) \to 0$ as $z \to \infty$ and that for all $z \ge y_0$ we have

$$|\pi(z) - \mathrm{li}(z)| \le \frac{z}{\log z} R(z) \tag{1.4}$$

and

$$\int_z^\infty \frac{|\pi(t) - \mathrm{li}(t)|}{t^2}\, dt \le C_0(y_0) R(z), \tag{1.5}$$

where $\mathrm{li}(z)$ is the logarithmic integral defined by

$$\mathrm{li}(z) := \int_0^z \frac{dt}{\log t}.$$

The classical version of the Prime Number Theorem allows us to take $R(z) = \exp(-c\sqrt{\log z})$ for some suitable constant $c > 0$. Using the zero-free region of Korobov and Vinogradov for the Riemann zeta-function, we obtain $R(z) = \exp(-c'(\log z)^{3/5}(\log\log z)^{-1/5})$ for some absolute constant $c' > 0$. If the Riemann Hypothesis holds, then one can take $R(z) = c'' z^{-1/2}\log^2 z$, where $c'' > 0$ is an absolute constant.

To state de Bruijn's result, we define

$$\mu_y(u) := \int_1^u y^{t-u}\omega(t)\,dt.$$

It is easy to see that $0 \le \mu_y(u)\log y \le 1 - y^{1-u}$ and that for every fixed $u \ge 1$, we have $\mu_y(u)\log y \to \omega(u)$ as $y \to \infty$. Precise expansions for $\mu_y(u)$ in terms of the powers of $\log y$ can be found in [13, Theorem III.6.18]. When $1 \le u \le 2$, the change of variable $t = \log v/\log y$ shows that

$$\mu_y(u)x = \int_1^u t^{-1}y^t\,dt = \int_y^x \frac{dv}{\log v} = \mathrm{li}(x) - \mathrm{li}(y).$$

Since $\Phi(x, y) = \pi(x) - \pi(y) + 1$ when $1 \le u \le 2$, (1.4) clearly implies that

$$\Phi(x, y) = \mu_y(u)x + (\pi(x) - \mathrm{li}(x)) - (\pi(y) - \mathrm{li}(y)) + 1 = \mu_y(u)x + O\left(\frac{xR(y)}{\log y}\right).$$

It can be shown using (1.4) and (1.5) that

$$\prod_{p\le y}\left(1 - \frac{1}{p}\right) = \frac{e^{-\gamma}}{\log y}\left(1 + O(R(y))\right).$$

Thus we have, equivalently,

$$\Phi(x, y) = \mu_y(u)e^{\gamma}x\log y\prod_{p\le y}\left(1 - \frac{1}{p}\right) + O\left(\frac{xR(y)}{\log y}\right). \tag{1.6}$$

Essentially, de Bruijn [4] showed that this formula holds uniformly for $x \ge y \ge y_0$. In Section 3 we shall derive an explicit version of (1.6), which will be applied in Section 4 to obtain numerically explicit estimates with suitable y_0 and $R(y)$. Our main results are summarized in the following theorem.

Theorem 1.1. *Uniformly for $x \ge y \ge 2$, we have*

$$\left|\Phi(x, y) - \mu_y(u)e^{\gamma}x\log y\prod_{p\le y}\left(1 - \frac{1}{p}\right)\right| < 4.403611\frac{x}{(\log y)^{3/4}}\exp\left(-\sqrt{\frac{\log y}{6.315}}\right).$$

Conditionally on the Riemann Hypothesis, we have

$$\left|\Phi(x,y) - \mu_y(u)e^{\gamma}x\log y\prod_{p\le y}\left(1-\frac{1}{p}\right)\right| < 0.449774\frac{x\log y}{\sqrt{y}}$$

uniformly for $x \ge y \ge 11$.

The following consequence of Theorem 1.1 is sometimes more convenient to use.

Corollary 1.2. *Uniformly for* $x \ge y \ge 2$, *we have*

$$|\Phi(x,y) - \mu_y(u)x| < 4.434084\frac{x}{(\log y)^{3/4}}\exp\left(-\sqrt{\frac{\log y}{6.315}}\right).$$

Conditionally on the Riemann Hypothesis, we have

$$|\Phi(x,y) - \mu_y(u)x| < 0.460680\frac{x\log y}{\sqrt{y}}$$

uniformly for $x \ge y \ge 11$.

2. Lower bounds for $\Phi(x,y)$

Before moving on to the derivation of Theorem 1.1, we prove a clean lower bound for $\Phi(x,y)$ which is applicable in a wide range. This lower bound, which is interesting in itself, will be used in the proof of Theorem 1.1 and Corollary 1.2 in Section 4. We start by proving the following result, which provides a numerically explicit lower bound for the implicit constant in the error term in (1.3). As we already mentioned, our method can easily be adapted to yield a numerically explicit upper bound as well, though it will not be needed in the present paper.

Proposition 2.1. *Define* $\Delta(x,y)$ *by*

$$\Phi(x,y) = \frac{x}{\log y}\left(\omega(u) + \frac{\Delta(x,y)}{\log y}\right)$$

for $2 \le y \le \sqrt{x}$. *Let* $y_0 = 602$. *For every positive integer* $k \ge 3$, *we define*

$$\Delta_k^- = \Delta_k^-(y_0) := \inf\left\{\min(\Delta(x,y),0) \colon y \ge y_0 \text{ and } 2 \le u < k\right\}.$$

Then $\Delta_3^- > -0.563528$, $\Delta_4^- > -0.887161$, *and* $\Delta_k^- > -0.955421$ *for all* $k \ge 5$.

Proof. Let $y_1 := 2{,}278{,}383$. Suppose first that $y \geq y_1$ and set

$$G(v) := \sum_{x^{1/v} < p \leq \sqrt{x}} \frac{1}{p}$$

for $2 \leq v \leq u$. By [5, Theorem 5.6],[1] we have

$$\left| G(v) - \log \frac{v}{2} \right| \leq \frac{c_1}{\log^2 y} \tag{2.1}$$

for all $y \geq y_1$, where $c_1 = 0.4/\log y_1$. We shall also make use of the following inequality [5, Corollary 5.2][2]:

$$\frac{z}{\log z}\left(1 + \frac{c_3}{\log z}\right) \leq \pi(z) \leq \frac{z}{\log z}\left(1 + \frac{c_2}{\log z}\right), \tag{2.2}$$

where $c_2 = 1 + 2.53816/\log y_1$ and $c_3 = 1 + 2/\log y_1$. We start with the range $2 \leq u \leq 3$. In this range, we have

$$\begin{aligned} \Phi(x,y) &= \#\{n \leq x\colon P^-(n) > y \text{ and } \Omega(n) \leq 2\} \\ &= \pi(x) - \pi(y) + 1 + \sum_{y < p \leq \sqrt{x}} \sum_{p \leq q \leq x/p} 1 \\ &= \pi(x) - \pi(y) + 1 + \sum_{y < p \leq \sqrt{x}} (\pi(x/p) - \pi(p) + 1), \end{aligned}$$

where $\Omega(n)$ denotes the total number of prime factors of n, with multiplicity counted. Since

$$\sum_{y < p \leq \sqrt{x}} (\pi(p) - 1) = \sum_{\pi(y) < j \leq \pi(\sqrt{x})} (j-1) = \frac{\pi(\sqrt{x})(\pi(\sqrt{x}) - 1)}{2} - \frac{\pi(y)(\pi(y) - 1)}{2},$$

we see that

$$\pi(x) - \pi(y) + 1 - \sum_{y < p \leq \sqrt{x}} (\pi(p) - 1) > \pi(x) - \frac{\pi(\sqrt{x})^2}{2} + \frac{\pi(\sqrt{x})}{2}.$$

It follows from (2.2) that

[1] In [2] it is claimed that the proof of [5, Theorem 4.2] is incorrect due to the application of an incorrect zero density estimate of Ramaŕe [10, Theorem 1.1]. In a footnote on p. 2299 of the same paper, the authors state that the bounds asserted in [5] are likely affected for this reason. However, since they also give a correct proof of [5, Theorem 4.2] (see [2, Corollary 11.2]), one verifies easily that the proof of [5, Theorem 5.6], which relies only on [5, Theorem 4.2], partial summation, and numerical computation, remains valid.

[2] For the same reason mentioned above, it is reasonable to suspect that the bounds given in [5, Corollary 5.2] are also affected. However, one can verify these bounds without much difficulty. Indeed, (5.2) of [5, Corollary 5.2] is superseded by [11, Corollary 1], while (5.3) and (5.4) of [5, Corollary 5.2] follow from [1, Lemmas 3.2–3.4] and direct calculations.

$$\Phi(x,y) > \frac{x}{\log x}\left(1+\frac{c_3}{\log x}\right) - \frac{x}{2\log^2\sqrt{x}}\left(1+\frac{c_2}{\log\sqrt{x}}\right)^2 + \frac{\sqrt{x}}{2\log\sqrt{x}} + \sum_{y<p\le\sqrt{x}} \pi(x/p). \tag{2.3}$$

To handle the sum in (2.3), we appeal to (2.2) again to arrive at

$$\sum_{y<p\le\sqrt{x}} \pi(x/p) \ge \sum_{y<p\le\sqrt{x}} \left(\frac{x}{p\log(x/p)} + \frac{c_3 x}{p\log^2(x/p)}\right).$$

By partial summation we see that

$$\sum_{y<p\le\sqrt{x}} \frac{1}{p\log(x/p)} = \frac{1}{\log x}\int_{2^-}^{u} \frac{v}{v-1}\,dG(v) = \frac{1}{\log y}\left(\frac{G(u)}{u-1} + \frac{1}{u}\int_{2^-}^{u} \frac{G(v)}{(v-1)^2}\,dv\right).$$

From (2.1) it follows that

$$\frac{G(u)}{u-1} \ge \frac{1}{u-1}\left(\log\frac{u}{2} - \frac{c_1}{\log^2 y}\right),$$

and

$$\begin{aligned}\int_{2^-}^{u} \frac{G(v)}{(v-1)^2}\,dv &\ge \int_{2}^{u} \frac{1}{(v-1)^2}\left(\log\frac{v}{2} - \frac{c_1}{\log^2 y}\right)dv \\ &= -\frac{1}{u-1}\log\frac{u}{2} + \int_2^u \frac{1}{v(v-1)}\,dv - \frac{c_1}{\log^2 y}\left(1-\frac{1}{u-1}\right) \\ &= -\frac{u}{u-1}\log\frac{u}{2} + \log(u-1) - \frac{c_1}{\log^2 y}\left(1-\frac{1}{u-1}\right).\end{aligned}$$

Hence

$$\begin{aligned}\sum_{y<p\le\sqrt{x}} \frac{x}{p\log(x/p)} &\ge \frac{x}{\log y}\left(\frac{\log(u-1)}{u} - \frac{2c_1}{u\log^2 y}\right) \\ &= \frac{x}{\log y}\left(\omega(u) - \frac{2c_1}{u\log^2 y}\right) - \frac{x}{\log x}.\end{aligned} \tag{2.4}$$

Similarly, we have

$$\begin{aligned}\sum_{y<p\le\sqrt{x}} \frac{1}{p\log^2(x/p)} &= \frac{1}{\log^2 x}\int_{2^-}^{u}\left(\frac{v}{v-1}\right)^2 dG(v) \\ &= \frac{1}{\log^2 x}\left(\frac{G(u)u^2}{(u-1)^2} + 2\int_{2^-}^{u} \frac{vG(v)}{(v-1)^3}\,dv\right).\end{aligned}$$

By (2.1) we have

$$\frac{G(u)u^2}{(u-1)^2} \geq \frac{u^2}{(u-1)^2}\left(\log\frac{u}{2} - \frac{c_1}{\log^2 y}\right),$$

and

$$\int_{2^-}^{u} \frac{vG(v)}{(v-1)^3}\,dv \geq \int_{2}^{u} \frac{v}{(v-1)^3}\left(\log\frac{v}{2} - \frac{c_1}{\log^2 y}\right) dv.$$

Since

$$\begin{aligned}\int_{2}^{u} \frac{v}{(v-1)^3}\log\frac{v}{2}\,dv &= -\left(\frac{1}{u-1} + \frac{1}{2(u-1)^2}\right)\log\frac{u}{2} + \int_{2}^{u}\left(\frac{1}{v-1} + \frac{1}{2(v-1)^2}\right)\frac{dv}{v} \\ &= -\frac{2u-1}{2(u-1)^2}\log\frac{u}{2} + \frac{1}{2}\int_{2}^{u}\left(\frac{1}{(v-1)^2} + \frac{1}{v(v-1)}\right) dv \\ &= -\frac{u^2}{2(u-1)^2}\log\frac{u}{2} + \frac{1}{2}\left(\log(u-1) + 1 - \frac{1}{u-1}\right)\end{aligned}$$

and

$$\int_{2}^{u} \frac{v}{(v-1)^3}\,dv = -\frac{2u-1}{2(u-1)^2} + \frac{3}{2},$$

we have

$$\sum_{y<p\leq\sqrt{x}} \frac{x}{p\log^2(x/p)} \geq \frac{x}{\log^2 x}\left(\log(u-1) + \frac{u-2}{u-1} - \frac{4c_1}{\log^2 y}\right). \tag{2.5}$$

Inserting (2.4) and (2.5) into (2.3) yields

$$\Delta(x,y) \geq g(u) - \frac{2c_1}{u\log y} + \frac{\log y}{uy^{3/2}} - \frac{1}{u^2}\left(2 - c_3 + \frac{4c_1c_3}{\log^2 y} + \frac{8c_2}{u\log y} + \frac{8c_2^2}{u^2\log^2 y}\right),$$

where

$$g(u) := \frac{c_3}{u^2}\left(\log(u-1) + \frac{u-2}{u-1}\right).$$

Using Mathematica we find that $\Delta_3^- > -0.301223$ when $y \geq y_1$.

Now we proceed to bound Δ_k^- for $k \geq 4$ recursively when $y \geq y_1$. Let $k \geq 3$ be arbitrary. It is easily seen that the following variant of Buchstab's identity (1.2) holds for any $z \in [y, x]$:

$$\Phi(x,y) = \Phi(x,z) + \sum_{y<p\le z} \Phi(x/p,p^-), \tag{2.6}$$

where $p^- < p$ is any real number sufficiently close to p. For $3 \le k \le u < k+1$ and $y \ge y_1$, we obtain by taking $z = x^{1/3}$ that

$$\Phi(x,y) = \Phi\left(x, x^{1/3}\right) + \sum_{y<p\le x^{1/3}} \Phi(x/p,p^-). \tag{2.7}$$

We have already shown that

$$\Phi\left(x,x^{1/3}\right) \ge \frac{x}{\log x^{1/3}}\left(\omega\left(\frac{\log x}{\log x^{1/3}}\right) + \frac{\Delta_3^-}{\log x^{1/3}}\right) = \frac{3x}{\log y}\left(\frac{\omega(3)}{u} + \frac{3\Delta_3^-}{u^2\log y}\right). \tag{2.8}$$

Note that $2 < \log(x/p)/\log(p^-) < k$. Thus, we have

$$\Phi(x/p,p^-) \ge \frac{x}{p\log(p^-)}\left(\omega\left(\frac{\log(x/p)}{\log(p^-)}\right) + \frac{\Delta_k^-}{\log(p^-)}\right).$$

Since $\omega(u)$ is continuous on $[1,\infty)$, it follows from (2.7) and (2.8) that

$$\Phi(x,y) \ge \frac{3x}{\log y}\left(\frac{\omega(3)}{u} + \frac{3\Delta_3^-}{u^2\log y}\right) + \sum_{y<p\le x^{1/3}} \frac{x}{p\log p}\left(\omega\left(\frac{\log x}{\log p}-1\right) + \frac{\Delta_k^-}{\log p}\right). \tag{2.9}$$

By partial summation we see that

$$\sum_{y<p\le x^{1/3}} \frac{1}{p\log^2 p} < \int_y^\infty \frac{1}{t\log^2 t}\,d\pi(t) = -\frac{\pi(y)}{y\log^2 y} + \int_y^\infty \frac{\log t + 2}{t^2\log^3 t}\pi(t)\,dt,$$

which, by (2.2), is

$$\begin{aligned}
&< -\frac{1}{\log^3 y}\left(1+\frac{c_3}{\log y}\right) + \int_y^\infty \frac{\log t+2}{t\log^4 t}\left(1+\frac{c_2}{\log t}\right)dt \\
&= -\frac{1}{\log^3 y}\left(1+\frac{c_3}{\log y}\right) + \frac{1}{2\log^2 y} + \frac{c_2+2}{3\log^3 y} + \frac{c_2}{2\log^4 y} \\
&= \frac{1}{\log^2 y}\left(\frac{1}{2} + \left(\frac{c_2}{3}-1\right)\frac{1}{\log y} + \left(\frac{c_2}{2}-c_3\right)\frac{1}{\log^2 y}\right) \\
&< \frac{1}{\log^2 y}\left(\frac{1}{2} + \left(\frac{c_2}{3}-1\right)\frac{1}{\log y}\right).
\end{aligned}$$

Hence

$$\sum_{y<p\le x^{1/3}} \frac{\Delta_k^- x}{p\log^2 p} \ge \frac{\Delta_k^- x}{\log^2 y}\left(\frac{1}{2}+\left(\frac{c_2}{3}-1\right)\frac{1}{\log y}\right). \tag{2.10}$$

On the other hand, we have

$$\begin{aligned}
&\sum_{y<p\le x^{1/3}} \frac{1}{p\log p}\omega\left(\frac{\log x}{\log p}-1\right)\\
&\quad = \frac{1}{\log x}\int_{3^-}^{u} v\omega(v-1)\,dG(v)\\
&\quad = \frac{1}{\log x}\left(\int_{3}^{u}\omega(v-1)\,dv + \int_{3^-}^{u} v\omega(v-1)\,d\left(G(v)-\log\frac{v}{2}\right)\right).
\end{aligned}$$

Observe that

$$\int_{3}^{u}\omega(v-1)\,dv = u\omega(u)-3\omega(3)$$

and that

$$\begin{aligned}
\int_{3^-}^{u} v\omega(v-1)\,d\left(G(v)-\log\frac{v}{2}\right) &= u\omega(u-1)\left(G(v)-\log\frac{v}{2}\right)-3\omega(2)\left(G(3)-\log\frac{3}{2}\right)\\
&\quad -\int_{3^-}^{u}\left(G(v)-\log\frac{v}{2}\right)\,d(v\omega(v-1)).
\end{aligned}$$

By [13, (6.23), p. 562] and [13, Theorems III.5.7 & III.6.6], we have, for all $v\ge 3$, that

$$\begin{aligned}
&\frac{d}{dv}(v\omega(v-1)) = \omega(v-2)+\omega'(v-1) \ge \frac{1}{2}-\rho(v-1) \ge \frac{1}{2}-\rho(2) = \log 2-\frac{1}{2},\\
&\frac{d}{dv}(v\omega(v-1)) \le 1+\rho(v-1) \le 1+\rho(2) = 2-\log 2,
\end{aligned}$$

where ρ is the Dickman-de Bruijn function defined to be the unique continuous solution to the delay differential equation $t\rho'(t)+\rho(t-1)=0$ for $t\ge 1$, subject to the initial value condition $\rho(t)=1$ for $0\le t\le 1$. Moreover, we have

$$\lim_{v\to 3^-}\frac{d}{dv}(v\omega(v-1)) = \lim_{v\to 3^-}(\omega(v-2)+\omega'(v-1)) = -\frac{1}{4}.$$

It follows by (2.1) that

$$\int_{3^-}^{u} \left(G(v) - \log\frac{v}{2}\right) d(v\omega(v-1)) \le \frac{c_1}{\log^2 y}\left(u\omega(u-1) - 3\omega(2)\right).$$

Thus we have

$$\int_{3^-}^{u} v\omega(v-1)\, d\left(G(v) - \log\frac{v}{2}\right) \ge -\frac{2c_1 u\omega(u-1)}{\log^2 y} \ge -\frac{2c_1 M_0 u}{\log^2 y},$$

where $M_0 = 0.5671432\ldots$. Hence we have shown that

$$\sum_{y<p\le x^{1/3}} \frac{x}{p\log p}\omega\left(\frac{\log x}{\log p} - 1\right) \ge \frac{x}{\log y}\left(\omega(u) - \frac{3\omega(3)}{u} - \frac{2c_1 M_0}{\log^2 y}\right). \tag{2.11}$$

Combining (2.9), (2.10) and (2.11), we deduce that

$$\Delta(x,y) \ge \frac{9\Delta_3^-}{u^2} + \frac{\Delta_k^-}{2} - \frac{1}{\log y}\left(2c_1 M_0 - \left(\frac{c_2}{3} - 1\right)\Delta_k^-\right)$$

for $k \le u < k+1$. Therefore, $\Delta_{k+1}^- \ge \min(\Delta_k^-, a_k^-)$ for all $k \ge 3$, where

$$a_k^- := \frac{9\Delta_3^-}{k^2} + \frac{\Delta_k^-}{2} - \frac{1}{\log y_1}\cdot \max\left(2c_1 M_0 - \left(\frac{c_2}{3} - 1\right)\Delta_k^-, 0\right).$$

Consequently, we have $\Delta_4^- > -0.451835$ and $\Delta_k^- > -0.480075$ for all $k \ge 5$.

Suppose now that $602 \le y \le y_1$. By [11, Theorem 20] we can replace (2.1) with

$$\left|G(v) - \log\frac{v}{2}\right| \le \frac{d_1}{\sqrt{y}\log y},$$

where $d_1 = 2$. Moreover, (2.2) remains true if we replace c_2 and c_3 by $d_2 = 1.2762$ and $d_3 = 1$, respectively, according to [5, Corollary 5.2]. With these changes, we run the same argument used to handle the case $y \ge y_1$ and get

$$\Delta(x,y) > g(u) - \frac{2d_1}{u\sqrt{y}} + \frac{\log y}{uy^{3/2}} - \frac{1}{u^2}\left(2 - d_3 + \frac{4d_1 d_3}{\sqrt{y}\log y} + \frac{8d_2}{u\log y} + \frac{8d_2^2}{u^2\log^2 y}\right)$$

when $2 \le u \le 3$ and

$$\Delta(x,y) \ge \frac{9\Delta_3^-}{u^2} + \frac{\Delta_k^-}{2} - \frac{1}{\log y}\left(\frac{2d_1 M_0 \log y}{\sqrt{y}} - \left(\frac{d_2}{3} - 1\right)\Delta_k^-\right)$$

when $3 \le k \le u < k+1$, so that we can take

$$a_k^- = \frac{9\Delta_3^-}{k^2} + \frac{\Delta_k^-}{2} - \frac{1}{\log y_0}\cdot \max\left(\frac{2d_1 M_0 \log y_0}{\sqrt{y_0}} - \left(\frac{d_2}{3} - 1\right)\Delta_k^-, 0\right).$$

As a consequence, we have $\Delta_3^- > -0.563528$, $\Delta_4^- > -0.887161$ and $\Delta_k^- > -0.955421$ for all $k \geq 5$. This completes the proof of the proposition. □

The next result provides a numerical lower bound for $\omega(u)$ on $[3, \infty)$.

Lemma 2.2. *We have* $\omega(u) > 0.549307$ *for all* $u \geq 3$.

Proof. Consider first the case $u \in [3, 4]$. Since $(t\omega(t))' = \omega(t-1)$ for $t \geq 2$ and $\omega(t) = (\log(t-1)+1)/t$ for $t \in [2, 3]$, we have

$$\omega(u) = \frac{1}{u}\left(\log 2 + 1 + \int_3^u \frac{\log(t-2)+1}{t-1}\,dt\right)$$

for $u \in [3, 4]$. Note that $u\omega'(u) = \omega(u-1) - \omega(u) = S(u)/u$, where

$$S(u) := \frac{u(\log(u-2)+1)}{u-1} - \log 2 - 1 - \int_3^u \frac{\log(t-2)+1}{t-1}\,dt.$$

Since

$$\begin{aligned} S'(u) &= \frac{1}{u-1}\left(\log(u-2) + 1 + \frac{u}{u-2} - \frac{u(\log(u-2)+1)}{u-1} - (\log(u-2)+1)\right) \\ &= \frac{u(1-(u-2)\log(u-2))}{(u-2)(u-1)^2}, \end{aligned}$$

we know that $S(u)$ is strictly increasing on $[3, u_1]$ and strictly decreasing on $[u_1, 4]$, where $u_1 = 3.7632228...$ is the unique solution to the equation $(u-2)\log(u-2) = 1$. But $S(3) = 1/2 - \log 2 < 0$ and

$$S(4) = \frac{\log 2 + 1}{3} - \int_3^4 \frac{\log(t-2)+1}{t-1}\,dt > 0.$$

It follows that $S(u)$ has a unique zero $u_2 \in [3, 4]$. The numerical value of u_2 is given by $u_2 = 3.4697488...$, according to Mathematica. Hence $S(u) < 0$ for $u \in [3, u_2)$ and $S(u) > 0$ for $u \in (u_2, 4]$. The same is true for $\omega'(u)$, which implies that $\omega(u)$ is strictly decreasing on $[3, u_2]$ and strictly increasing on $[u_2, 4]$. Thus, $\omega(u) \geq \omega(u_2) = 0.5608228...$ for $u \in [3, 4]$.

Consider now the case $u \in [4, \infty)$. It is known [8] that $\omega(t)$ satisfies

$$|\omega(t) - e^{-\gamma}| \leq \frac{\rho(t-1)}{t}$$

for all $t \geq 1$. Since $\rho(t)$ is strictly decreasing on $[4, \infty)$, we have $\omega(u) \geq e^{-\gamma} - \rho(3)/4$ for all $u \geq 4$. To find the value of $\rho(3)$, we use $t\rho'(t) + \rho(t-1) = 0$ for $t \geq 1$ and $\rho(t) = 1 - \log t$ for $t \in [1, 2]$ to obtain

$$\rho(u) = 1 - \log 2 - \int_2^u \frac{1 - \log(t-1)}{t}\, dt$$

for $u \in [2, 3]$. It follows that

$$\omega(u) \geq e^{-\gamma} - \frac{1}{4}\left(1 - \log 2 - \int_2^3 \frac{1 - \log(t-1)}{t}\, dt\right) = 0.5493073...$$

for all $u \geq 4$. We have therefore shown that $\omega(u) > 0.549307$ for all $u \geq 3$. $\square$

We are now ready to prove the following clean lower bound for $\Phi(x, y)$ that we alluded to.

Theorem 2.3. *We have $\Phi(x, y) > 0.4x/\log y$ uniformly for all $7 \leq y \leq x^{2/3}$.*

Proof. In the range $\max(7, x^{2/5}) \leq y \leq x^{2/3}$, we have trivially $\Phi(x, y) \geq \pi(x) - \pi(y) + 1$. By [5, Corollary 5.2] we have

$$\begin{aligned}
\pi(x) - \pi(y) &\geq \frac{x}{\log x}\left(1 + \frac{1}{\log x}\right) - \frac{y}{\log y}\left(1 + \frac{1.2762}{\log y}\right) \\
&= \left(\frac{1}{u}\left(1 + \frac{1}{\log x}\right) - \frac{y}{x}\left(1 + \frac{1.2762u}{\log x}\right)\right)\frac{x}{\log y} \\
&> \left(\frac{2}{5}\left(1 + \frac{1}{\log x}\right) - \frac{1}{x^{1/3}}\left(1 + \frac{3.1905}{\log x}\right)\right)\frac{x}{\log y} > 0.4\frac{x}{\log y}
\end{aligned}$$

whenever $x \geq 41{,}217$. Furthermore, we have verified $\Phi(x, y) > 0.4x/\log y$ for $\max(7, x^{2/5}) \leq y \leq x^{2/3}$ with $x \leq 41{,}217$ using Mathematica. Hence, $\Phi(x, y) > 0.4x/\log y$ holds in the range $\max(7, x^{2/5}) \leq y \leq x^{2/3}$.

Consider now the case $\max(x^{1/3}, 7) \leq y \leq x^{2/5}$. Following the proof of Proposition 2.1, we have

$$\begin{aligned}
\Phi(x, y) &= \pi(x) - \pi(y) + 1 + \sum_{y < p \leq \sqrt{x}} (\pi(x/p) - \pi(p) + 1) \\
&= \pi(x) - M(x, y) + \sum_{y < p \leq x^{1/2}} \pi(x/p), \qquad (2.12)
\end{aligned}$$

where

$$M(x,y) := \frac{1}{2}\pi\left(\sqrt{x}\right)^2 - \frac{1}{2}\pi\left(\sqrt{x}\right) - \frac{1}{2}\pi(y)^2 + \frac{3}{2}\pi(y) - 1.$$

To handle the sum in (2.12), we appeal to Theorem 5 and its corollary from [11] to arrive at

$$G(v) - \log\frac{v}{2} > -\frac{1}{2\log^2\sqrt{x}} - \frac{1}{\log^2 y} \geq -\frac{33}{25\log^2 y}$$

in the range $\max(x^{1/3}, 7) \leq y \leq x^{2/5}$. By [11, Corollary 1] we have

$$\sum_{y<p\leq x^{1/2}} \pi(x/p) > x \sum_{y<p\leq x^{1/2}} \frac{1}{p\log(x/p)} = \frac{x}{\log x}\int_{2^-}^{u} \frac{v}{v-1}\,dG(v),$$

provided that $x \geq 289$. The right-hand side of the above can be estimated in the same way as in the proof of Proposition 2.1, so we obtain

$$\sum_{y<p\leq\sqrt{x}} \pi(x/p) > \frac{x}{\log y}\left(\omega(u) - \frac{66}{25u\log^2 y}\right) - \frac{x}{\log x}.$$

On the other hand, we see by [5, Corollary 5.2] and [11, Corollary 2] that

$$\pi(x) - M(x,y) > \pi(x) - \frac{1}{2}\pi\left(\sqrt{x}\right)^2 \geq \frac{x}{\log x}\left(1 + \frac{1}{\log x}\right) - \frac{25x}{8\log^2 x} = \frac{x}{\log x} - \frac{17x}{8\log^2 x}$$

for $x \geq 114^2$. Collecting the estimates above and using the inequality $\omega(u) \geq \omega(5/2) = 2(\ln(3/2)+1)/5$ for $u \in [5/2, 3]$, we find that

$$\Phi(x,y) > \frac{\omega(5/2)x}{\log y} - \frac{17x}{8\log^2 x} - \frac{66x}{25u\log^3 y} \geq \frac{\omega(5/2)x}{\log y} - \frac{17x}{50\log^2 y} - \frac{132x}{125\log^3 y} > 0.4\frac{x}{\log y}$$

for all $\max(46, x^{1/3}) \leq y \leq x^{2/5}$. For $x^{1/3} \leq y \leq x^{2/5}$ with $7 \leq y \leq 46$, we have verified the inequality $\Phi(x,y) > 0.4x/\log y$ directly through numerical computation.

Next, we consider the range $7 \leq y < x^{1/3}$. By Proposition 2.1 and Lemma 2.2 we have

$$\Phi(x,y) > \frac{x}{\log y}\left(0.549307 - \frac{0.955421}{\log y}\right) > 0.4\frac{x}{\log y},$$

provided that $y \geq 602$. To deal with the range $7 \leq y \leq \min(x^{1/3}, 602)$, we follow the inclusion-exclusion technique used in [7, Section 3]. For any integer $n \geq 1$, let $\nu(n)$ denote the number of distinct prime factors of n. We start by "pre-sieving" with the primes 2, 3, and 5: for any $x \geq 1$ the number of integers $n \leq x$ with $\gcd(n, 30) = 1$ is $(4/15)x + r_x$, where $|r_x| \leq 14/15$. Let $P_5(y)$ be the product of the primes in $(5, y]$. Then we have by the Bonferroni inequalities that

$$\Phi(x,y) \geq \sum_{\substack{d|P_5(y)\\ \nu(d)\leq 3}} \mu(d)\left(\frac{4}{15}\cdot\frac{x}{d} + r_{x/d}\right) \geq a(y)x - b(y),$$

where

$$a(y) := \frac{4}{15}\sum_{\substack{d|P_5(y)\\ \nu(d)\leq 3}} \frac{\mu(d)}{d} = \frac{4}{15}\sum_{j=0}^{3}(-1)^j \sum_{\substack{d|P_5(y)\\ \nu(d)=j}} \frac{1}{d},$$

$$b(y) := \frac{14}{15}\sum_{j=0}^{3}\binom{\pi(y)-3}{j}.$$

By Newton's identities, the inner sum in the definition of $a(y)$ can be represented in terms of the power sums of $1/p$ over all primes $5 < p \leq y$. Thus, we have $\Phi(x,y) > 0.4x/\log y$ whenever $a(y) > 0.4/\log y$ and $x > b(y)/(a(y) - 0.4/\log y)$. Using Mathematica, we find that the inequality $\Phi(x,y) > 0.4x/\log y$ holds for $7 \leq y \leq 602$ and $x \geq 13{,}160{,}748$. Finally, we have verified the inequality $\Phi(x,y) > 0.4x/\log y$ directly for $7 \leq y \leq x^{1/3}$ with $x \leq 13{,}160{,}748$ by numerical calculations, completing the proof of our theorem. □

Remark 2.1. Note that for $y \in [5, 7)$ we have

$$\Phi(x,y) \geq \frac{4}{15}x - \frac{14}{15} > 0.4\frac{x}{\log 5} \geq 0.4\frac{x}{\log y},$$

provided that $x \geq 52$. Combined with Theorem 2.3 and numerical examination of the case $11 \leq x \leq 52$, this implies that the inequality $\Phi(x,y) > 0.4x/\log y$ holds in the slightly larger range $5 \leq y \leq x^{2/3}$ if one assumes $x \geq 41$.

3. An explicit version of de Bruijn's estimate

To prove Theorem 1.1, we shall first develop an explicit version of (1.6) with a general $R(y)$, following [4], where $R(y)$ is a positive decreasing function satisfying the same conditions described in the introduction. Suppose that $y_0 \geq 3$. For each $z \geq 2$, put

$$Q(z) := \prod_{p\leq z}\left(1 - \frac{1}{p}\right).$$

We start by estimating $Q(y)$ for $y \geq y_0$. Using a Stieltjes integral, we may write

$$\log\frac{Q(z)}{Q(y)} = \int_y^z \log\left(1 - t^{-1}\right)\, d\,\mathrm{li}(y) + \int_y^z \log\left(1 - t^{-1}\right)\, d(\pi(y) - \mathrm{li}(t)), \tag{3.1}$$

where $z \geq y \geq y_0$. The first integral on the right-hand side of the above is equal to

$$\int_y^z \log\left(1-t^{-1}\right)\frac{dt}{\log t} = -\log\frac{\log z}{\log y} + \int_y^z \left(t^{-1}+\log\left(1-t^{-1}\right)\right)\frac{dt}{\log t}.$$

Since

$$-\frac{1}{2t(t-1)} < t^{-1}+\log\left(1-t^{-1}\right) < 0$$

for all $t \geq y_0$, we have

$$-\frac{1}{2}\int_y^\infty \frac{dt}{t(t-1)\log t} < \int_y^z \left(t^{-1}+\log\left(1-t^{-1}\right)\right)\frac{dt}{\log t} < 0.$$

But a change of variable shows that

$$\int_y^\infty \frac{dt}{t(t-1)\log t} = \int_1^\infty \frac{dt}{t(y^t-1)} \leq \frac{1}{y-1}\int_1^\infty \frac{dt}{t^2} = \frac{1}{y-1},$$

where we have used the inequality $y^t - 1 \geq (y-1)t$ for $t \geq 1$ and $y \geq y_0$. It follows that

$$-\frac{1}{2(y-1)} \leq \int_y^z \log\left(1-t^{-1}\right)\,d\,\mathrm{li}(y) + \log\frac{\log z}{\log y} < 0. \tag{3.2}$$

Now we estimate the second integral on the right-hand side of (3.1). By (1.4) and partial integration we have

$$\left|\int_y^z \log\left(1-t^{-1}\right)\,d(\pi(y)-\mathrm{li}(t))\right|$$

$$\leq \log\left(1-y^{-1}\right)^{-1}\frac{y}{\log y}R(y) + \log\left(1-z^{-1}\right)^{-1}\frac{z}{\log z}R(z) + \int_y^z \frac{|\pi(t)-\mathrm{li}(t)|}{t(t-1)}\,dt.$$

Using (1.5) we see that

$$\int_y^z \frac{|\pi(t)-\mathrm{li}(t)|}{t(t-1)}\,dt \leq \frac{C_0(y_0)y_0}{y_0-1}R(y).$$

It is clear that the function

$$\log\left(1-t^{-1}\right)^{-1}\frac{t}{\log t} = \frac{1}{\log t}\sum_{n=0}^{\infty}\frac{t^{-n}}{n+1}$$

is strictly decreasing for $t \in (1, \infty)$. Since $R(t)$ is decreasing on $[y_0, \infty)$, we find that

$$\left| \int_y^z \log\left(1 - t^{-1}\right) \, d(\pi(y) - \mathrm{li}(t)) \right| \le \left(2 \log\left(1 - y_0^{-1}\right)^{-1} \frac{y_0}{\log y_0} + \frac{C_0(y_0) y_0}{y_0 - 1} \right) R(y).$$

Combining this inequality with (3.1) and (3.2) yields

$$-C_2(y_0) R(y) \le \log \frac{Q(z)}{Q(y)} + \log \frac{\log z}{\log y} \le C_1(y_0) R(y) \tag{3.3}$$

for $z \ge y \ge y_0$, where

$$C_1(y_0) = 2 \log\left(1 - y_0^{-1}\right)^{-1} \frac{y_0}{\log y_0} + \frac{C_0(y_0) y_0}{y_0 - 1},$$
$$C_2(y_0) = C_1(y_0) + \sup_{t \ge y_0} \frac{1}{2(t-1) R(t)}.$$

Exponentiating (3.3) we obtain

$$-C_4(y_0) R(y) \le \frac{Q(z) \log z}{Q(y) \log y} - 1 \le C_3(y_0) R(y) \tag{3.4}$$

for $z \ge y \ge y_0$, where

$$C_3(y_0) = \sup_{t \ge y_0} \frac{\exp(C_1(y_0) R(t)) - 1}{R(t)} = \frac{\exp(C_1(y_0) R(y_0)) - 1}{R(y_0)},$$
$$C_4(y_0) = \sup_{t \ge y_0} \frac{1 - \exp(-C_2(y_0) R(t))}{R(t)} = C_2(y_0).$$

As a consequence, we have by letting $z \to \infty$ in (3.4) and using the fact that $Q(z) \log z \to e^{-\gamma}$ as $z \to \infty$, that

$$e^{\gamma} \log y (1 - C_4(y_0) R(y)) \le \frac{1}{Q(y)} \le e^{\gamma} \log y (1 + C_3(y_0) R(y)). \tag{3.5}$$

Similarly, we derive from (3.3) that

$$\frac{e^{-\gamma}}{\log y}(1 - C_6(y_0) R(y)) \le Q(y) \le \frac{e^{-\gamma}}{\log y}(1 + C_5(y_0) R(y)) \tag{3.6}$$

for $y \ge y_0$, where

$$C_5(y_0) = \sup_{t \geq y_0} \frac{\exp(C_2(y_0)R(t)) - 1}{R(t)} = \frac{\exp(C_2(y_0)R(y_0)) - 1}{R(y_0)},$$

$$C_6(y_0) = \sup_{t \geq y_0} \frac{1 - \exp(-C_1(y_0)R(t))}{R(t)} = C_1(y_0).$$

For $x \geq y \geq 2$, we define

$$\psi(x, y) := \frac{\Phi(x, y)}{xQ(y)}.$$

We then need to estimate $\eta(x, y) = \psi(x, y) - \lambda(x, y)$, where $\lambda(x, y) := e^\gamma \mu_y(u) \log y$. For $1 \leq u \leq 2$ this can be done straightforward. Indeed, we have $\Phi(x, y) = \pi(x) - \pi(y) + 1$ and $\omega(u) = 1/u$ when $1 \leq u \leq 2$, so that

$$\eta(x, y) = \frac{\pi(x) - \pi(y) + 1}{xQ(y)} - e^\gamma \log y \int_1^u t^{-1} y^{t-u}\, dt.$$

Note that

$$\left| \pi(x) - \pi(y) - x \int_1^u t^{-1} y^{t-u}\, dt \right| = \left| \pi(x) - \pi(y) - \int_y^x \frac{dt}{\log t} \right| \leq \left(\frac{x}{\log x} + \frac{y}{\log y} \right) R(y).$$

From (3.5) it follows that $|\eta(x, y)| \leq e^\gamma \alpha_y(u) R(y)$ for $y \geq y_0$ and $u \in [1, 2]$, where

$$\alpha_y(u) := \frac{\log y}{y^u R(y)} + C_3(y_0) \left(\frac{\log y}{y^u} + \log y \int_1^u t^{-1} y^{t-u}\, dt \right) + (1 + C_3(y_0)R(y)) \left(\frac{1}{u} + y^{1-u} \right).$$

Integration by parts enables us to write

$$\log y \int_1^u t^{-1} y^{t-u}\, dt = \frac{1}{u} - y^{1-u} + \int_1^u t^{-2} y^{t-u}\, dt$$

for $y \geq y_0$. Hence $|\eta(x, y)| \leq e^\gamma \eta_1(y) R(y)$ for $y \geq y_0$ and $u \in [1, 2]$, where

$$\eta_1(y) := \sup_{t \geq y} \frac{\log t}{tR(t)} + \max_{u \in [1,2]} \left(C_3(y_0) I_y(u) + (1 + C_3(y_0)R(y)) \left(\frac{1}{u} + y^{1-u} \right) \right) \tag{3.7}$$

with

$$I_y(u) := \frac{1}{u} + \int_1^u t^{-2} y^{t-u}\, dt.$$

We remark that $I_y(u)$ is strictly decreasing on $[1,2]$ and hence satisfies $I_y(u) < 1$ for $u \in (1,2]$, since its derivative is

$$I'_y(u) = -\int_1^u t^{-2} y^{t-u} \log y \, dt < 0.$$

Thus, (3.7) simplifies to

$$\eta_1(y) = \sup_{t \geq y} \frac{\log t}{tR(t)} + C_3(y_0) + 2(1 + C_3(y_0)R(y)). \tag{3.8}$$

Suppose now that $y \geq y_0$ and $u \geq 2$. From (2.6) it follows that

$$\psi(x,y) = \psi(x,z)\frac{Q(z)}{Q(y)} + \sum_{y<p\leq z} \psi(x/p, p^-) \cdot \frac{1}{p} \prod_{y<q<p} \left(1 - \frac{1}{q}\right), \tag{3.9}$$

where $z \geq y \geq y_0$. Put $h := \log z / \log y \geq 1$ and

$$H_y(v) := \sum_{y<p\leq y^v} \frac{1}{p} \prod_{y<q<p} \left(1 - \frac{1}{q}\right) \tag{3.10}$$

for $v \geq 1$. Then we have $H_y(v) = 1 - Q(y^v)/Q(y)$. By partial summation, we see that (3.9) becomes

$$\psi(x,y) = \psi(y^u, y^h)(1 - H_y(h)) + \int_1^h \psi(y^{u-v}, (y^v)^-) \, dH_y(v). \tag{3.11}$$

By (3.4) we have

$$|H_y(v) - 1 + v^{-1}| \leq C_7(y_0)R(y),$$

where $C_7(y_0) := \max(C_3(y_0), C_4(y_0))$. Thus, one can think of $1 - v^{-1}$ as a smooth approximation to $H_y(v)$. Since we also expect $\lambda(x,y)$ to be a smooth approximation to $\psi(x,y)$, in view of (3.11) it is reasonable to expect

$$E_1(h; y, u) := \lambda(y^u, y) - \lambda(y^u, y^h)h^{-1} - \int_1^h \lambda(y^{u-v}, y^v) v^{-2} \, dv$$

to be small in size as a function of y. This can be easily verified when $1 \leq h \leq u/2$. Following de Bruijn [4], we have

$$\frac{\partial}{\partial h} E_1(h; y, u) = -h^{-1} \cdot \frac{\partial}{\partial h} \lambda(y^u, y^h) + h^{-2}\lambda(y^u, y) - h^{-2}\lambda(y^{u-h}, y^h). \tag{3.12}$$

Since

$$\frac{\lambda(y^u, y^h)}{e^\gamma \log y} = h \int_1^{u/h} y^{ht-u}\omega(t)\, dt,$$

we find

$$\frac{\partial}{\partial h}\left(\frac{\lambda(y^u, y^h)}{e^\gamma \log y}\right) = \int_1^{u/h} y^{ht-u}\omega(t)\, dt + h\left(\log y \int_1^{u/h} y^{ht-u}(t\omega(t))\, dt - uh^{-2}\omega(uh^{-1})\right).$$

Recall that $(t\omega(t))' = \omega(t-1)$ for $t \in \mathbb{R}$ with the obvious extension $\omega(t) = 0$ for $t < 1$. It follows that

$$\begin{aligned}\log y \int_1^{u/h} y^{ht-u}(t\omega(t))\, dt &= h^{-1}y^{ht-u}(t\omega(t))\Big|_1^{u/h} - h^{-1}\int_1^{u/h} y^{ht-u}\omega(t-1)\, dt \\ &= uh^{-2}\omega(uh^{-1}) - h^{-1}y^{h-u} - h^{-1}y^h \int_1^{u/h-1} y^{ht-u}\omega(t)\, dt \\ &= uh^{-2}\omega(uh^{-1}) - h^{-1}y^{h-u} - \left(h^2 e^\gamma \log y\right)^{-1}\lambda(y^{u-h}, y^h).\end{aligned}$$

Hence we have

$$\begin{aligned}\frac{\partial}{\partial h}\lambda(y^u, y^h) &= e^\gamma \log y \left(\int_1^{u/h} y^{ht-u}\omega(t)\, dt - y^{h-u}\right) - h^{-1}\lambda(y^{u-h}, y^h) \\ &= h^{-1}\lambda(y^u, y^h) - e^\gamma y^{h-u}\log y - h^{-1}\lambda(y^{u-h}, y^h).\end{aligned}$$

Inserting this in (3.12) yields

$$\frac{\partial}{\partial h}E_1(h; y, u) = h^{-1}e^\gamma y^{h-u}\log y.$$

Integrating both sides with respect to h and using the initial value condition $E_1(1; y, u) = 0$, we obtain

$$E_1(h; y, u) = e^\gamma \log y \int_1^h t^{-1}y^{t-u}\, dt < e^\gamma y^{h-u}. \tag{3.13}$$

In what follows, we shall always suppose that $1 \le h \le u/2$. Following de Bruijn [4], we proceed to show that

$$E_3(h; y, u) := \lambda(y^u, y) - \lambda(y^u, y^h)(1 - H(h)) - \int_1^h \lambda(y^{u-v}, y^v)\, dH(h)$$

is small in size as a function of y. This is intuitive, because

$$\lambda(y^u, y^h)h^{-1} - \int_1^h \lambda(y^{u-v}, y^v)v^{-2}\, dv \tag{3.14}$$

is a good approximation to $\lambda(y^u, y)$, as we have already demonstrated. Consequently, the expression (3.14) can be thought of as a smooth approximation to

$$\lambda(y^u, y^h)(1 - H(h)) - \int_1^h \lambda(y^{u-v}, y^v)\, dH(h).$$

Moreover, we have by (3.11) that

$$\eta(x, y) = \eta(y^u, y^h)(1 - H_y(h)) + \int_1^h \eta(y^{u-v}, (y^v)^-)\, dH_y(v) - E_3(h; y, u), \tag{3.15}$$

which will later be used to estimate $\eta(x, y)$. To estimate $E_3(h; y, u)$, let us write $E_3(h; y, u) = E_1(h; y, u) + E_2(h; y, u)$, where

$$E_2(h; y, u) := -\int_1^h \lambda(y^{u-v}, y^v)\, d\left(H(v) - 1 + v^{-1}\right) + (H(h) - 1 + h^{-1})\lambda(y^u, y^h).$$

Then we expect $E_2(h; y, u)$ to be small in size as a function of y. Using (3.10) and the observation that $H(1) = 0$, we have

$$|E_2(h; y, u)| \leq \left(\left|\lambda(y^u, y^h) - \lambda(y^{u-h}, y^h)\right| + \int_1^h \left| \frac{\partial}{\partial v}\lambda(y^{u-v}, y^v) \right| dv \right) C_7(y_0)R(y). \tag{3.16}$$

Note that

$$\begin{aligned} \frac{\lambda(y^u, y^h) - \lambda(y^{u-h}, y^h)}{he^\gamma \log y} &= \int_1^{u/h} y^{ht-u}\omega(t)\, dt - \int_2^{u/h} y^{ht-u}\omega(t-1)\, dt \\ &= \int_1^2 y^{ht-u}\omega(t)\, dt + \int_2^{u/h} y^{ht-u}(\omega(t) - \omega(t-1))\, dt \end{aligned}$$

$$= \int_1^2 t^{-1} y^{ht-u}\, dt - \int_2^{u/h} y^{ht-u} t\omega'(t)\, dt.$$

By Theorems III.5.7 and III.6.6 in [13] we have

$$|\omega'(t)| \le \rho(t) \le \frac{1}{\Gamma(t+1)} \tag{3.17}$$

for all $t \ge 1$. It follows that

$$\left|\lambda(y^u, y^h) - \lambda(y^{u-h}, y^h)\right| \le he^\gamma \log y \left(\int_1^2 t^{-1} y^{ht-u}\, dt + \int_2^{u/h} y^{ht-u} t\rho(t)\, dt \right). \tag{3.18}$$

This inequality will later be used in conjunction with the formulas

$$h \log y \int_1^2 t^{-1} y^{ht-u}\, dt = \frac{y^{2h-u}}{2} - y^{h-u} + \int_1^2 t^{-2} y^{ht-u}\, dt \tag{3.19}$$

and

$$\begin{aligned} h \log y \int_2^{u/h} y^{ht-u} t\rho(t)\, dt &= uh^{-1}\rho(uh^{-1}) - 2\rho(2) y^{2h-u} - \int_2^{u/h} y^{ht-u} (t\rho(t))'\, dt \\ &\le uh^{-1}\rho(uh^{-1}) - 2\rho(2) y^{2h-u} + \int_2^{u/h} y^{ht-u} \rho(t-1)\, dt. \end{aligned} \tag{3.20}$$

On the other hand, we have

$$\frac{\lambda(y^{u-v}, y^v)}{e^\gamma \log y} = v \int_2^{u/v} y^{vt-u} \omega(t-1)\, dt,$$

which implies that

$$\frac{\partial}{\partial v} \left(\frac{\lambda(y^{u-v}, y^v)}{e^\gamma \log y} \right) = \int_2^{u/v} y^{vt-u} (1 + tv \log y) \omega(t-1)\, dt - uv^{-1} \omega(uv^{-1} - 1).$$

By partial integration, the right side of the above is easily seen to be

$$-2y^{2v-u} - \int_2^{u/v} y^{vt-u} t\omega'(t-1)\, dt.$$

Hence, we arrive at

$$\int_1^h \left|\frac{\partial}{\partial v}\lambda(y^{u-v}, y^v)\right| dv \le e^\gamma \log y \left(2\int_1^h y^{2v-u}\, dv + \int_1^h \int_2^{u/v} y^{vt-u} t|\omega'(t-1)|\, dtdv\right).$$

Furthermore, we have by Fubini's theorem that

$$\int_1^h \int_2^{u/v} y^{vt-u} t|\omega'(t-1)|\, dtdv = \int_2^{u/h}\int_1^h y^{vt-u} t|\omega'(t-1)|\, dv\, dt + \int_{u/h}^{u}\int_1^{u/t} y^{vt-u} t|\omega'(t-1)|\, dv\, dt,$$

the right side of which is easily seen to be

$$\frac{1}{\log y}\left(\int_2^{u/h} y^{ht-u}|\omega'(t-1)|\, dt + \int_{u/h}^{u} |\omega'(t-1)|\, dt - \int_2^u y^{t-u}|\omega'(t-1)|\, dt\right).$$

It follows that

$$\int_1^h \left|\frac{\partial}{\partial v}\lambda(y^{u-v}, y^v)\right| dv < e^\gamma \left(y^{2h-u} + \int_2^{u/h} y^{ht-u}|\omega'(t-1)|\, dt + \int_{u/h}^{u} |\omega'(t-1)|\, dt\right). \tag{3.21}$$

This estimate together with (3.18) will lead us to a good estimate for $E_2(h; y, u)$.

Now we derive estimates for $E_3(h; y, u)$ that suit our needs. Suppose that $k \le u < k+1$ and take $h = h_k = u/k$, where $k \ge 2$ is a positive integer. We first consider the case $k = 2$. In view of (3.19), we see that (3.18) simplifies to

$$\left|\lambda\left(y^u, y^{h_2}\right) - \lambda\left(y^{u-h_2}, y^{h_2}\right)\right| < e^\gamma \left(\frac{1}{2} + \int_1^2 t^{-2} y_0^{t-2}\, dt\right) = e^\gamma I_{y_0}(2)$$

for $y \ge y_0$. By (3.21) we have

$$\int_1^{h_2} \left|\frac{\partial}{\partial v}\lambda(y^{u-v}, y^v)\right| dv \le e^\gamma \left(1 + \int_2^3 |\omega'(t-1)|\, dt\right) = \frac{3e^\gamma}{2},$$

since $\omega'(t) = -1/t^2$ for $t \in [1, 2)$. Combining these estimates with (3.13) and (3.16), we obtain $E_3(h_2; y, u) \le e^\gamma \xi_2(y_0) R(y)$ for $y \ge y_0$ and $2 \le u < 3$, where

$$\xi_2(y_0) := \max_{t \ge y_0} \frac{1}{tR(t)} + C_7(y_0)\left(I_{y_0}(2) + \frac{3}{2}\right).$$

Now we handle the case $k \geq 3$. From (3.17)–(3.20) it follows that

$$\left|\lambda\left(y^u, y^{h_k}\right) - \lambda\left(y^{u-h_k}, y^{h_k}\right)\right| < e^\gamma \left(\frac{1}{\Gamma(k)} + \left(2\log 2 - \frac{3}{2}\right) y^{2-k} + \int_1^2 t^{-2} y^{t-k}\, dt + \int_2^3 y^{t-k}(1-\log(t-1))\, dt + \int_3^k y^{t-k} \frac{dt}{\Gamma(t)}\right),$$

where we have used the fact that $\rho(t) = 1 - \log t$ for $t \in [1,2]$. By (3.17) and (3.21) we have

$$\int_1^{h_k} \left|\frac{\partial}{\partial v}\lambda(y^{u-v}, y^v)\right| dv \leq e^\gamma \left(y^{2-k} + \int_2^3 y^{t-k} \frac{dt}{(t-1)^2} + \int_3^k y^{t-k}\frac{dt}{\Gamma(t)} + \int_k^{k+1} \frac{dt}{\Gamma(t)}\right).$$

Together with (3.13) and (3.16), these inequalities imply that $E_3(h_k; y, u) \leq e^\gamma \xi_k(y_0) R(y)$ for $y \geq y_0$ and $3 \leq k \leq u < k+1$, where

$$\xi_k(y_0) := \left(\max_{t \geq y_0} \frac{1}{tR(t)}\right) y_0^{2-k} + C_7(y_0) \left(\frac{1}{(k-1)!} + \int_k^{k+1} \frac{dt}{\Gamma(t)} + \left(2\log 2 - \frac{1}{2}\right) y_0^{2-k} + \int_1^2 t^{-2} y_0^{t-k}\, dt + \int_2^3 y_0^{t-k}\left(1 - \log(t-1) + \frac{1}{(t-1)^2}\right) dt + 2\int_3^k y_0^{t-k}\frac{dt}{\Gamma(t)}\right).$$

As a direct corollary, we obtain

$$\sum_{k=2}^{\infty} \xi_k(y_0) = \frac{y_0}{y_0 - 1} \max_{t \geq y_0} \frac{1}{tR(t)} + C_7(y_0)\left(e - \frac{1}{2} + \int_3^\infty \frac{dt}{\Gamma(t)} + \frac{1}{y_0 - 1}\left(2\log 2 - 1 + y_0 I_{y_0}(2) + \int_2^3 y_0^{t-2}\left(1 - \log(t-1) + \frac{1}{(t-1)^2}\right) dt + 2\int_3^\infty y_0^{\{t\}} \frac{dt}{\Gamma(t)}\right)\right),$$

where we have applied partial summation to derive

$$\sum_{k=3}^{\infty} \int_3^k y_0^{t-k} \frac{dt}{\Gamma(t)} = \left(\sum_{k=3}^{\infty} y_0^{-k}\right) \int_3^\infty y_0^t \frac{dt}{\Gamma(t)} - \int_3^\infty \left(\sum_{3 \leq k \leq t} y_0^{-k}\right) y_0^t \frac{dt}{\Gamma(t)}$$
$$= \frac{y_0^{-3}}{1 - y_0^{-1}} \int_3^\infty y_0^t \frac{dt}{\Gamma(t)} - \int_3^\infty \frac{y_0^{t-3}(1 - y_0^{-\lfloor t \rfloor + 2})}{1 - y_0^{-1}} \cdot \frac{dt}{\Gamma(t)}$$

$$= \frac{1}{y_0 - 1} \int_3^\infty y_0^{\{t\}} \frac{dt}{\Gamma(t)}.$$

For computational purposes, we can transform the last integral above by observing that

$$\int_3^\infty y_0^{\{t\}} \frac{dt}{\Gamma(t)} = \int_0^1 \left(\sum_{n=0}^\infty \frac{1}{(t+2)\cdots(t+2+n)} \right) y_0^t \frac{dt}{\Gamma(t+2)}.$$

Let

$$\gamma(s,z) := \int_0^z v^{s-1} e^{-v}\, dv$$

be the lower incomplete gamma function, where $s \in \mathbb{C}$ with $\Re(s) > 0$ and $z \geq 0$. It is well known that

$$\gamma(s,z) = z^s e^{-z} \sum_{n=0}^\infty \frac{z^n}{s(s+1)\cdots(s+n)},$$

from which it follows that

$$\sum_{n=0}^\infty \frac{1}{(t+2)\cdots(t+2+n)} = \gamma(t+2,1)e.$$

Thus we obtain

$$\begin{aligned}\sum_{k=2}^\infty \xi_k(y_0) = \frac{y_0}{y_0-1} \max_{t \geq y_0} \frac{1}{tR(t)} + C_7(y_0) \Bigg(e - \frac{1}{2} + \int_3^\infty \frac{dt}{\Gamma(t)} \\ + \frac{1}{y_0 - 1} \Bigg(2\log 2 - 1 + y_0 I_{y_0}(2) + \int_2^3 y_0^{t-2} \left(1 - \log(t-1) + \frac{1}{(t-1)^2} \right) dt \\ + 2e \int_0^1 y_0^t \frac{\gamma(t+2,1)}{\Gamma(t+2)}\, dt \Bigg) \Bigg). \end{aligned} \tag{3.22}$$

In Mathematica, the function $\gamma(t+2,1)$ can be evaluated by "Gamma[t+2,0,1]".

Finally, we are ready to estimate $\eta(x,y)$. Let

$$\eta_k(y) := \frac{1}{e^\gamma R(y)} \sup_{\substack{u \in [k,k+1) \\ t \geq y}} |\eta(t^u, t)|$$

for $k \geq 1$ and $y \geq y_0$, where the value of $\eta_1(y)$ is provided by (3.8). Using (3.15) and the estimates for $E_3(h_k; y, u)$ with $y \geq y_0$ and $2 \leq k \leq u < k+1$, we find

$$\eta_k(y) \leq \eta_{k-1}(y) + \xi_k(y_0)$$

for all $k \geq 2$ and $y \geq y_0$, from which we derive

$$\eta_k(y) \leq \eta_1(y) + \sum_{\ell=2}^{k} \xi_\ell(y_0)$$

for all $k \geq 1$ and $y \geq y_0$. Since $\eta_1(y)$ is decreasing on $[y_0, \infty)$, we have therefore shown that

$$|\eta(x,y)| \leq e^\gamma \left(\eta_1(y_0) + \sum_{k=2}^{\infty} \xi_k(y_0) \right) R(y) \tag{3.23}$$

for all $y \geq y_0$, where the infinite sum can be evaluated using (3.22). To derive an explicit version of de Bruijn's result (1.6), we observe that (3.6), (3.23) and [11, Theorem 23] imply that $Q(y)|\eta(x,y)| \leq C_8(y_0)R(y)/\log y$ for all $y \geq y_0$, where

$$C_8(y_0) := \beta(y_0) \left(\eta_1(y_0) + \sum_{k=2}^{\infty} \xi_k(y_0) \right)$$

with

$$\beta(y_0) := \begin{cases} 1, & \text{if } 3 \leq y_0 < 10^8, \\ \exp(C_2(y_0)R(y_0)), & \text{if } y_0 \geq 10^8. \end{cases}$$

Hence, it follows that

$$\left| \Phi(x,y) - \mu_y(u) e^\gamma x \log y \prod_{p \leq y} \left(1 - \frac{1}{p}\right) \right| < \frac{C_8(y_0) x R(y)}{\log y} \tag{3.24}$$

for all $y \geq y_0$.

4. Deduction of Theorem 1.1 and Corollary 1.2

Now we apply (3.24) to obtain explicit estimates for $\Phi(x,y)$ with specific choices of $R(y)$. Unconditionally, it has been shown [9, Corollary 2] that

$$|\pi(z) - \mathrm{li}(z)| \leq 0.2593 \frac{z}{(\log z)^{3/4}} \exp\left(-\sqrt{\frac{\log z}{6.315}} \right)$$

for all $z \geq 229$. With $y_0 \geq 229$, the function

$$R(z) = 0.2593(\log z)^{1/4} \exp\left(-\sqrt{\frac{\log z}{6.315}}\right)$$

is strictly decreasing on $[y_0, \infty)$ and satisfies (1.4) and (1.5) with

$$C_0(y_0) = 2\sqrt{\frac{6.315}{\log y_0}},$$

since

$$\begin{aligned}\int_z^\infty \frac{1}{t(\log t)^{3/4}} \exp\left(-\sqrt{\frac{\log t}{6.315}}\right) dt &= 2\int_{\sqrt{\log z}}^\infty \frac{1}{\sqrt{t}} \exp\left(-\frac{t}{\sqrt{6.315}}\right) dt \\ &< \frac{2}{(\log z)^{1/4}} \int_{\sqrt{\log z}}^\infty \exp\left(-\frac{t}{\sqrt{6.315}}\right) dt \\ &= \frac{2\sqrt{6.315}}{(\log z)^{1/4}} \exp\left(-\sqrt{\frac{\log z}{6.315}}\right)\end{aligned}$$

for $z \geq y_0$. Numerical computation using Mathematica allows us to conclude that

$$\left|\Phi(x,y) - \mu_y(u)e^\gamma x \log y \prod_{p \leq y}\left(1 - \frac{1}{p}\right)\right| < 4.403611 \frac{x}{(\log y)^{3/4}} \exp\left(-\sqrt{\frac{\log y}{6.315}}\right) \quad (4.1)$$

for all $x \geq y \geq 229$. Suppose now that $2 \leq y < 229$. Using the inequalities $\Phi(x,y) < x/\log y$ [6, Theorem], $\prod_{p\leq y}(1 - 1/p) < e^{-\gamma}/\log y$ [11, Theorem 23] and $0 \leq \mu_y(u) < 1/\log y$, we have

$$\left|\Phi(x,y) - \mu_y(u)e^\gamma x \log y \prod_{p \leq y}\left(1 - \frac{1}{p}\right)\right| < \frac{2x}{\log y} < 4.403611 \frac{x}{(\log y)^{3/4}} \exp\left(-\sqrt{\frac{\log y}{6.315}}\right)$$

for all $2 \leq y < 229$. Combining this with (4.1) proves the first half of Theorem 1.1.

Under the assumption of the Riemann Hypothesis, it is known [12, Corollary 1] that

$$|\pi(z) - \mathrm{li}(z)| < \frac{1}{8\pi}\sqrt{z}\log z$$

for all $z \geq 2657$. With $y_0 = 2657$ and

$$R(z) = \frac{\log^2 z}{8\pi\sqrt{z}},$$

Table 1
Numerical Constants.

constants	unconditional estimates		conditional estimates	
y_0	229	10^8	2657	10^8
$R(y_0)$	.156576	.097363	.047992	.001351
$C_0(y_0)$	2.156096	1.171019	.317985	.120362
$C_1(y_0)$	2.534430	1.279593	.571800	.228936
$C_2(y_0)$	2.548436	1.279593	.575723	.228940
$C_3(y_0)$	3.110976	1.362717	.579718	.228971
$C_4(y_0)$	2.548436	1.279593	.575723	.228940
$C_5(y_0)$	3.131827	1.362717	.583750	.228975
$C_6(y_0)$	2.534430	1.279593	.571800	.228936
$C_7(y_0)$	3.110976	1.362717	.579718	.228971
$C_8(y_0)$	16.982691	9.079975	4.638553	2.967998
$\eta_1(y_0)$	6.236726	3.628074	2.697198	2.229726
$\sum_{k=2}^{\infty} \xi_k(y_0)$	10.745960	4.388310	1.941356	.737355

we have

$$\int_z^\infty \frac{|\pi(t) - \mathrm{li}(t)|}{t^2}\, dt \le \frac{1}{8\pi} \int_z^\infty \frac{\log t}{t^{3/2}}\, dt = \frac{\log z + 2}{4\pi\sqrt{z}} \le C_0(y_0) R(z)$$

for $z \ge y_0$, where

$$C_0(y_0) = \frac{2(\log y_0 + 2)}{\log^2 y_0}.$$

Therefore, we conclude by (3.24) and numerical calculations that

$$\left| \Phi(x,y) - \mu_y(u) e^\gamma x \log y \prod_{p \le y} \left(1 - \frac{1}{p}\right) \right| < 0.184563 \frac{x \log y}{\sqrt{y}} \tag{4.2}$$

for all $x \ge y \ge 2657$. The values of relevant constants are recorded in Table 1.

To complete the proof of the second half of Theorem 1.1, it remains to deal with the case $11 \le y \le 2657$. For simplicity of notation we set

$$D(x,y) := \Phi(x,y) - \mu_y(u) e^\gamma x \log y \prod_{p \le y} \left(1 - \frac{1}{p}\right).$$

Using Mathematica we find that

$$M := \max_{11 \le z \le 2657} \frac{\mathrm{li}(z) - \pi(z)}{\sqrt{z}\log z} < 0.259141,$$

$$m := \min_{11 \le z \le 2657} e^\gamma \log z \prod_{p \le z} \left(1 - \frac{1}{p}\right) > 0.876248.$$

If $\sqrt{x} \le y < x$, then

$$\Phi(x, y) = \mu_y(u)x + (\pi(x) - \text{li}(x)) - (\pi(y) - \text{li}(y)) + 1.$$

Note that $x \le y^2 < 10^8$. Since $\pi(z) < \text{li}(z)$ for $2 \le z \le 10^8$ by [11, Theorem 16] and

$$\prod_{p \le z} \left(1 - \frac{1}{p}\right) < \frac{e^{-\gamma}}{\log z}$$

for $0 < z \le 10^8$ by [11, Theorem 23], we have

$$\begin{aligned} |D(x, y)| &< (1 - m)\left(1 - y^{-1}\right) \frac{x}{\log y} + M\sqrt{x} \log x + 1 \\ &\le \left((1 - m)\left(1 - y^{-1}\right) + M \frac{\log^2 y}{\sqrt{y}} + \frac{\log y}{y} \right) \frac{x}{\log y}, \end{aligned} \tag{4.3}$$

where we have used the fact that $\log x/\sqrt{x}$ is strictly decreasing on $[e^2, \infty)$. Numerical computation shows that the right side of (4.3) is $< 0.449774x \log y/\sqrt{y}$ for $11 \le y \le 2657$. Suppose now that $11 \le y \le \sqrt{x}$. By [7, Theorem 1], Theorem 2.3 and [11, Theorem 23] we have, for $11 \le y \le 2657$,

$$\begin{aligned} D(x, y) &\le \left(0.6 - \frac{m}{2}\left(1 - y^{-1}\right)\right) \frac{x}{\log y} < 0.449774 \frac{x \log y}{\sqrt{y}}, \\ D(x, y) &> (0.4 - M_0) \frac{x}{\log y} > -0.449774 \frac{x \log y}{\sqrt{y}}. \end{aligned}$$

This settles the case $11 \le y \le 2657$ and completes the proof of Theorem 1.1.

The proof of Corollary 1.2 is similar, and we shall only sketch it. When $y \ge y_0$, where $y_0 = 229$ for the unconditional estimate and $y_0 = 2657$ for the conditional estimate, we have by the triangle inequality that

$$|\Phi(x, y) - \mu_y(u)x| < |D(x, y)| + \left| 1 - e^\gamma \log y \prod_{p \le y} \left(1 - \frac{1}{p}\right) \right| \frac{x}{\log y}.$$

Then we bound $|D(x, y)|$ using the values of $C_8(y_0)$ listed in the table above. To estimate the second term, we use (3.6) when $y \ge 10^8$ and the inequality

$$m(y) < e^\gamma \log y \prod_{p \le y} \left(1 - \frac{1}{p}\right) < 1$$

when $y_0 \le y \le 10^8$, where $m(y)$ is given by

$$m(y) := \begin{cases} 0.983296, & \text{if } 229 \le y \le 2657, \\ 0.996426, & \text{if } 2657 \le y < 210{,}000, \\ 0.999643, & \text{if } 210{,}000 \le y \le 10^8, \end{cases}$$

according to [11, Theorem 23] and Mathematica. This leads to the asserted bounds for $y \geq y_0$. Suppose now that $y \leq y_0$. In this case, the proof of the unconditional bound is exactly the same as that of the unconditional bound in Theorem 1.1. As for the conditional bound, we argue in the same way as in the proof of Theorem 1.1 to get

$$|\Phi(x, y) - \mu_y(u)x| \leq \left(M \frac{\log^2 y}{\sqrt{y}} + \frac{\log y}{y}\right) \frac{x}{\log y}$$

when $\sqrt{x} \leq y < x$ and

$$|\Phi(x, y) - \mu_y(u)x| \leq \left(0.6 - \frac{1}{2}\left(1 - y^{-1}\right)\right) \frac{x}{\log y},$$
$$|\Phi(x, y) - \mu_y(u)x| > (0.4 - M_0) \frac{x}{\log y},$$

when $11 \leq y \leq \sqrt{x}$. Together, these inequalities yield the asserted conditional bound.

Remark 4.1. The bounds in Theorem 1.1 and its corollary may be improved. For example, the numerical values of the sum $\sum_{k=2}^{\infty} \xi_k(y_0)$ may be reduced by keeping ρ (or even $|\omega'|$) in all of the relevant integrals, but of course the computational complexity is expected to increase as a cost. In addition, our method would allow an extension of the range $x \geq y \geq 11$ in the second half of Theorem 1.1 to the entire range $x \geq y \geq 2$ if we argue with $y_0 = 2657$ replaced by some smaller value and enlarge the constant 0.449774.

Data availability

Data will be made available on request.

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