LCM Products & k-Colossally Abundant Numbers

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1. The LCM Convolution

Def^n 1. Let $f, g : \mathbb{N} \to \mathbb{C}$ be arithmetic functions. Then the LCM convolution of f with g is defined by

$$(f \times g)(n) := \sum_{[a,b]=n} f(a)\, g(b).$$

For any $k \in \mathbb{N}$, we write

$$f^k(n) = (\underbrace{f \times \cdots \times f}_{k \text{ times}})(n)$$

Rmk 1. $f^0 \equiv 1$. For $k \in \mathbb{N}$, we have

$$f^k(n) = \sum_{[a_1,\cdots,a_k]=n} f(a_1)\cdots f(a_k)$$

$$= \sum_{a_1,\cdots,a_k | n} \sum_{d | \frac{n}{[a_1,\cdots,a_k]}} \mu(d) f(a_1)\cdots f(a_k)$$

$$= \sum_{d|n} \mu(d) \sum_{a_1,\cdots,a_k | \frac{n}{d}} f(a_1)\cdots f(a_k)$$

$$= \sum_{d|n} \mu(d) F\left(\frac{n}{d}\right)^k,$$

where $F(n) := \sum_{d|n} f(d)$.

Def^n 2. For $k \in \mathbb{R}$, the kth LCM product of an arithmetic function of $f : \mathbb{N} \to \mathbb{C}$ is defined by

$$f^k(n) := \sum_{d|n} \mu(d) F\left(\frac{n}{d}\right)^k,$$

provided $F(n) := \sum_{d|n} f(d) > 0$ for all n.

E.g 1. Let $f_s(n) = n^s$, where $s \in \mathbb{C}$. Then

$$f_s^k(n) = \sum_{d|n} \mu(d)\, \sigma_s\left(\frac{n}{d}\right)^k,$$

where $\sigma_s(n) := \sum_{d|n} d^s$. In particular, $\sigma^{[k]}(n) := f_1(n)$.

E.g. 2 $\varphi^k(n) = \sum_{d|n} \mu(d) \left(\frac{n}{d}\right)^k$ (Jordan totient function)

Rmk 2. If f is multiplicative, i.e., $f(mn) = f(m) f(n)$ whenever $(m,n) = 1$, then so is f^k, because F^k is. In particular,

$$\sigma^{[k]}(n) = \prod_{p^a \| n} \sigma^{[k]}(p^a) = \prod_{p^a \| n} \frac{(p^{a+1}-1)^k - (p^a-1)^k}{(p-1)^k} \approx \sigma(n)^k.$$

Thm 1. (Grönwall) $\limsup_{n\to\infty} \frac{\sigma(n)}{n \log\log n} = e^{\gamma}.$

Why do we care?

1. Selberg's sieve: Minimize the quadratic form

$$Q(\lambda) := \sum_{\substack{d_1, d_2 \leq z \\ d_1, d_2 \ \square\text{-free}}} \frac{\lambda_{d_1} \lambda_{d_2}}{[d_1, d_2]}$$

$$= \sum_{n \leq z^2} \frac{1}{n} \sum_{\substack{d_1, d_2 \leq z \\ [d_1, d_2] = n}} |\mu(d_1) \mu(d_2)| \lambda_{d_1} \lambda_{d_2}.$$

Formally, $\displaystyle\sum_{n=1}^{\infty} \frac{(f \times g)(n)}{n^s} = \sum_{a,b=1}^{\infty} \frac{f(a) g(b)}{[a,b]^s}$.

2. Let X be a scheme of finite type.

$$\zeta_X(s) = \prod_{p} \left(1 - N(x)^{-s}\right)^{-1}$$

$\zeta_{X \sqcup Y}(s) = \zeta_X(s) \zeta_Y(s)$.

$$\mathrm{blog}: (a(n)) \longmapsto ((\Omega \cdot a) * a^{*-1})(n)$$

$$\Omega_X(s) = \mathrm{blog}(\zeta_X(s)) = \sum_q \# X(\mathbb{F}_q)\, q^{-s}.$$

Then
$$\Omega_{X \sqcup Y}(s) = \Omega_X(s) + \Omega_Y(s),$$

$$\Omega_{X \times Y}(s) = \Omega_X(s) \cdot \Omega_Y(s).$$

Let $Z_X(s) = \zeta(s)^{-1}\Omega_X(s)$. Then

$$Z_{X \sqcup Y}(s) = Z_X(s) + Z_Y(s),$$

$$Z_{X \times Y}(s) = (Z_X \times Z_Y)(s) = \sum_{n,m=1}^{\infty} \frac{a_X(m)\, b_Y(n)}{[m,n]^s}.$$

Scheme theory & Rankin-Selberg in arithmetic geometry.

tensor product of two automorphic representations yields the pointwise product of the associated Dirichlet series.

Pointwise product of Dirichlet series.

Thm 2. (F.-Kobayashi-Molnar) For any $k>1$, we have

$$\limsup_{n\to\infty} \frac{\sigma^{[k]}(n)}{(n\log\log n)^k} = \frac{e^{k\gamma}}{\zeta(k)},$$

where ζ is the Riemann zeta-function.

2. Colossally Abundant Numbers and Ramanujan's Thm

Def^n 3. A positive integer N is called a colossally abundant number

if $\exists\ \varepsilon>0$ s.t. $\dfrac{\sigma(N)}{N^{1+\varepsilon}} \geq \dfrac{\sigma(n)}{n^{1+\varepsilon}}$ for all $n>1$.

Rmk 3. Grönwall's thm $\Rightarrow$ $\forall\ \varepsilon>0$, $\exists$ a colossally abundant number.

Let $F(x,a) := \dfrac{1}{\log x} \log \dfrac{x^{a+1}-1}{x^{a+1}-x}$, where $x>1$ and $a>0$, and set

$F(x,0) := \infty$. Then F is strictly decreasing as x and a increase.

Prop 1. N is colossally abundant for ε iff for every $p^a \| N$ one has

$$F(p, a+1) \leq \varepsilon \leq F(p, a).$$

Def^n 4. For each prime p, let $E_p := \{ F(p,a) : a \geq 1 \}$. Set $E := \bigcup_p E_p = \{ \varepsilon_i \}_{i=1}^{\infty} \downarrow$. $\varepsilon \in E$ is called a critical epsilon value. For every fixed $\varepsilon > 0$, define a strictly decreasing sequence $\{ x_l \}_{l=1}^{\infty}$ by $F(x_l, l) = \varepsilon$.

Conjecture (Robin) The sets E_p are pairwise disjoint.

Prop 2. Let $\varepsilon \in (0, \infty) \setminus E$. Then $\exists!$ colossally abundant number N for ε. Moreover, if $p^{\alpha_p} \| N$, then $\alpha_p = \lfloor a_p \rfloor$, where a_p is the unique solution to $F(p, a) = \varepsilon$. Thus $\alpha_p = l$ for $x_{l+1} < p < x_l$ and $\alpha_p = 0$ if $p > x_1$.

Def^n 5. For every $\varepsilon > 0$, let N_ε be the largest CA number for ε.

Facts: (a) N_ε is constant on $[\varepsilon_i, \varepsilon_{i-1})$ and distinct, $\varepsilon_0 = \infty$.

So $\{N_\varepsilon\}_{\varepsilon > 0} = \{N_i\}_{i=1}^{\infty} \nearrow$, where $N_i = N_{\varepsilon_i}$.

(b) $N_i = \prod_{l \geq 1} \prod_{x_{l+1} < p \leq x_l} p^l$.

(c) For $N_i \leq n \leq N_{i+1}$, $G(n) \leq \max(G(N_i), G(N_{i+1}))$, where

$$G(n) := \frac{\sigma(n)}{n \log\log n},$$

Thm 3 (Ramanujan) RH $\Rightarrow$ $G(n) < e^\gamma$ for all sufficiently large n.

Proof Sketch: Assume RH and $G(n) \geq e^\gamma$ for arbitrarily large n

- Fact (c) $\Rightarrow$ $G(N) \geq e^\gamma$ for arbitrarily large $N = N_\varepsilon$.

- Fact (b) $\Rightarrow$ $$\frac{\sigma(N)}{N} = \prod_{i\geq 1} \prod_{x_{i+1}<p\leq x_i} \left(1+\frac{1}{p}+\cdots+\frac{1}{p^i}\right) \leq \frac{\prod_{x_2<p\leq x_1}\left(1-\frac{1}{p^2}\right)}{\prod_{p\leq x_1}\left(1-\frac{1}{p}\right)}.$$

- $x_2 < \sqrt{2x_1}$ $\Rightarrow$ $$\frac{\sigma(N)}{N} \leq \frac{\prod_{\sqrt{2x_1}<p\leq x_1}\left(1-\frac{1}{p^2}\right)}{\prod_{p\leq x_1}\left(1-\frac{1}{p}\right)}$$

- $$\prod_{\sqrt{2x_1}<p\leq x_1}\left(1-\frac{1}{p^2}\right) = \exp\left(-\frac{\sqrt{2}}{\sqrt{x_1}\log x_1} + O\left(\frac{1}{\sqrt{x_1}\log^2 x_1}\right)\right).$$

RH $\quad \theta(x) = x + O(\sqrt{x}(\log x)^2)$

- RH $\Rightarrow$ $$\prod_{p\leq x_1}\left(1-\frac{1}{p}\right)^{-1} \leq e^{\gamma}\log\theta(x_1)\exp\left(\frac{2+\beta}{\sqrt{x_1}\log x_1} + O\left(\frac{1}{\sqrt{x_1}\log^2 x_1}\right)\right)$$

where $$\beta = 2\sum_{\rho:\, 0<\operatorname{Re}\rho<1} \rho^{-1} \overset{RH}{=} 2\sum_{\rho=\frac{1}{2}+it,\, t>0} |\rho|^{-2} = \sum_{\rho} |\rho|^{-2} = \gamma + 2 - \log 2\pi = 0.0461\ldots.$$

$$\theta(x) = \sum_{p\leq x} \log p.$$

- RH $\Rightarrow$ $$\log\theta(x_1) \leq \log\log N \exp\left(-\frac{0.986\sqrt{2}}{\sqrt{x_1}\log x_1} + \frac{0.484}{\sqrt{x_1}\log^3 x_1}\right) \text{ for } x_1 \geq 20{,}000.$$

- $$\frac{\sigma(N)}{N\log\log N} \leq e^{\gamma}\cdot\exp\left(-\frac{c}{\sqrt{x_1}\log x_1} + O\left(\frac{1}{\sqrt{x_1}\log^2 x_1}\right)\right) \text{ for sufficiently large } x_1,$$

where $c = 1.986\sqrt{2} - (2+\beta) > 0$, a contradiction.

3. k-Colossally Abundant Numbers and An Analogue of Ramanujan's Thm

Def^n 6. Let $\varepsilon > 0$ and $k > 0$. A positive integer N is called a k-colossally abundant number for ε if $\dfrac{\sigma^{[k]}(N)}{N^{(1+\varepsilon)k}} \geq \dfrac{\sigma^{[k]}(n)}{n^{(1+\varepsilon)k}}$ for all $n > 1$.

Recall : $$F(p,a) = \frac{1}{\log p} \log \frac{\sigma(p^a)}{p\,\sigma(p^{a-1})} = \frac{1}{\log p} \log \frac{p^{a+1}-1}{p^{a+1}-p}.$$

Want $$F^{[k]}(p,a) = \frac{1}{\log p^k} \log \frac{\sigma^{[k]}(p^a)}{p^k \sigma^{[k]}(p^{a-1})} = \frac{1}{k \log p} \log \frac{(p^{a+1}-1)^k - (p^a-1)^k}{(p^{a+1}-p)^k - (p^a-p)^k}.$$

So we define $$F^{[k]}(x,a) = \frac{1}{k \log x} \log \frac{(x^{a+1}-1)^k - (x^a-1)^k}{(x^{a+1}-x)^k - (x^a-x)^k}.$$

Thm 4 (F.-Kobayashi-Molnar) Let $k > 1$, $x > 1$ and $a \geq 1$. Then

(i) $F^{[k]}(x,a)$ is strictly decreasing as x and a increase and is strictly increasing as k increases.

(ii) $\lim_{x\to\infty} F^{[k]}(x,a) = 0$ and $\lim_{x\to 1^+} F^{[k]}(x,a) = +\infty$.

(iii) $\lim_{k\to\infty} F^{[k]}(x,a) = F(x,a)$.

This theorem allows us to define, $\forall \varepsilon > 0$ & $k > 1$, the strictly decreasing sequence $\{x_l^{[k]}\}_{l=1}^{\infty}$ by $F^{[k]}(x_l^{[k]}, l) = \varepsilon$.

By a similar approach, we are able to prove the following analogue of Ramanujan's thm.

Thm 5 (F.-Kobayashi-Molnar) Let $G^{[k]}(n) = \dfrac{\sigma^{[k]}(n)}{(n\log\log n)^k}$ and assume RH. Then for every $k > \frac{3}{2}$, there exists $n_k \in \mathbb{N}$ s.t. $G^{[k]}(n) < \dfrac{e^{k\gamma}}{\zeta(k)}$ for all $n \geq n_k$.

4. Future Work

Q 1. Does Thm 5 still hold when $1 < k \le \frac{3}{2}$?

Thm 6 (Ramanujan–Robin) RH $\Leftrightarrow$ $G(n) < e^{\gamma}$ for all $n > 5040$.

Q 2: What's the correct analogue of Thm 6 ?

Conjecture. There exists a function $\rho : (\frac{3}{2}, \infty) \longrightarrow \mathbb{N}$ s.t. RH holds iff for every $k > \frac{3}{2}$, $G^{[k]}(n) < \dfrac{e^{k\gamma}}{\zeta(k)}$ for all $n > \rho(k)$.

(What is $\rho(k)$ precisely ?)

Thm 7 (Lagarias) RH $\Leftrightarrow$ $\sigma(n) < H_n e^{H_n} \log H_n$, $\forall n \ge 1$. Here $H_n = \sum_{k=1}^{n} \frac{1}{k}$.

Q 3. What's the correct analogue of Thm 7 ?

Appendix: The first 15 CA numbers

n	factorization of n	$\sigma(n)/n$
2	2	1.500
6	$2 \cdot 3$	2.000
12	$2^2 \cdot 3$	2.333
60	$2^2 \cdot 3 \cdot 5$	2.800
120	$2^3 \cdot 3 \cdot 5$	3.000
360	$2^3 \cdot 3^2 \cdot 5$	3.250
2520	$2^3 \cdot 3^2 \cdot 5 \cdot 7$	3.714
5040	$2^4 \cdot 3^2 \cdot 5 \cdot 7$	3.838
55440	$2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$	4.187
720720	$2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13$	4.509
1441440	$2^5 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13$	4.581
4324320	$2^5 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$	4.699
21621600	$2^5 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$	4.855
367567200	$2^5 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17$	5.141
6983776800	$2^5 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19$	5.412
160626866400	$2^5 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$	5.647
321253732800	$2^6 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$	5.692
9316358251200	$2^6 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29$	5.888
288807105787200	$2^6 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31$	6.078
2021649740510400	$2^6 \cdot 3^3 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31$	6.187
6064949221531200	$2^6 \cdot 3^4 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31$	6.238
224403121196654400	$2^6 \cdot 3^4 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 37$	6.407

Conjecture: The ratio of consecutive CA numbers is a prime.

Thm 2.2 (Lehmer) The map

$$u : (\mathbb{R}^{\mathbb{N}}, +, \times, 0, \delta) \to (\mathbb{R}^{\mathbb{N}}, +, \cdot, 0, 1)$$

$$f \mapsto F(n)$$

is an isomorphism with inverse μ.