PNT: From Classical to Pretentious

Gauss & Legendre: $\pi(x) := \#\{p \leq x : p \text{ prime}\} \approx \frac{x}{\log x}$

$$\pi(x) \sim \frac{x}{\log x} \quad \text{as } x \to \infty.$$

Hadamard & de la Vallée Poussin: PNT holds.

Riemann: $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$, $s = \sigma + it$, $\sigma > 1$.

- $\zeta(s)$ can be meromorphically continued to $\mathbb{C}$ with a simple pole at $s = 1$.

- $\underbrace{\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s)}_{,} = \pi^{-\frac{1-s}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s)$.

- $\xi(s) := \frac{1}{2} \underset{\Delta}{s} \underset{\square}{(s-1)} \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s)$ has $\xi(s) = e^{A+Bs} \prod_{\rho} \left(1 - \frac{s}{\rho}\right) e^{\frac{s}{\rho}}$.

$$\Lambda(n) := \begin{cases} \log p & n = p^m \\ 0 & \text{otherwise} \end{cases}$$

PNT $\Leftrightarrow$ $\psi(x) := \sum_{n \leq x} \Lambda(n) \sim x$ ✓

1. Classical Method : complex analysis, analytic continuation, functional equation, Hadamard product, explicit formula

$$\psi(x) = x - \sum_{\substack{\rho \\ 0<\mathrm{Re}\,\rho<1}} \frac{x^\rho}{\rho} - \underbrace{\frac{1}{2}\log(1-x^2)}$$

If $\mathrm{Re}\,\rho < 1$, $\sum_\rho \frac{x^\rho}{\rho}$ smaller than x.

2. Elementary Method :

e.g. Erdős & Selberg : Use calculus.

$$\psi(x)\log x + \sum_{n \leq x} \Lambda(n)\psi\left(\frac{x}{n}\right) = 2x\log x + O(x)$$

3. Others

- Probabilistic Method

- Pretentious Approach: multiplicative functions f: • $f(1) = 1$
  • $f(mn) = f(m) f(n)$, $(m,n) = 1$.

Classical Method

$\underline{\zeta(1+it) \neq 0 \text{ for all } t \in \mathbb{R}} \iff PNT$

Pf that $\zeta(1+it) \neq 0$: $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$, $\sigma > 1$

$$\log \zeta(s) = \sum_p \log\left(1 - \frac{1}{p^s}\right)^{-1} = \sum_{p^m} \frac{1}{m} \cdot p^{-ms}, \quad \sigma > 1$$

$$= \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^s \log n}$$

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}.$$

$$4 \cdot \log|\zeta(s)| = 4 \operatorname{Re} \log \zeta(s) = 4 \cdot \sum_{p^m} \frac{1}{m} p^{-m\sigma} \underline{\cos(mt\log p)} \quad ①$$

Inequality: $3 + 4\cos\theta + \overset{2\cos^2\theta - 1}{\widetilde{\cos 2\theta}} = 2(1+\cos\theta)^2 \geq 0$. ?

$$1 \cdot \log|\zeta(\sigma + i \cdot 2t)| = 1 \cdot \sum_{p^m} \frac{1}{m} p^{-m\sigma} \cos(2mt\log p) \quad ②$$

$$3 \log \zeta(\sigma) = 3 \sum_{p^m} \frac{1}{m} p^{-m\sigma} \quad ③$$

$$\Rightarrow 3\log\zeta(\sigma) + 4\log|\zeta(\sigma+it)| + \log|\zeta(\underline{\sigma + i \cdot 2t})| = \sum_{p^m} \underline{\frac{1}{m} p^{-m\sigma}} \Big[\overbrace{3 + 4\cos(mt\log p) + \cos(2mt\log p)}^{\geq 0} \Big]$$

$$\geq 0, \quad \sigma > 1, \ t \in \mathbb{R} \setminus \{0\}.$$

$$\Rightarrow \quad \zeta(\sigma)^3 \cdot |\zeta(\sigma+it)|^4 \cdot |\zeta(\sigma + i \cdot 2t)| \geq 1.$$

If $\zeta(1+it) = 0$, then $\zeta(\sigma) \sim \frac{1}{\sigma - 1}$, $\zeta(\sigma+it) \sim C \cdot (\sigma - 1)$, $\zeta(\sigma + i \cdot 2t) = O(1)$ as $\sigma \to 1^+$.

$LHS = O((\sigma - 1)) \to 0$ as $\sigma \to 1^+$, a contradiction. $\square$

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$$

$|p^{-it}| = 1$

$$\underset{\substack{\shortparallel \\ 0}}{\zeta(1+it)} \approx \underbrace{\prod_p \left(1 - \frac{1}{p} \cdot p^{-it}\right)^{-1}}_{\text{small when } p^{-it} \approx -1} \qquad 1 - \frac{1}{p} \cdot \underbrace{p^{-it}}_{-1} \text{ large}$$

$$\Downarrow$$

$$\zeta(1+i2t) \approx \prod_p \left(1 - \frac{1}{p} \cdot \underbrace{(p^{-it})^2}_{\approx 1}\right)^{-1} \approx \prod_p \left(1 - \frac{1}{p}\right)^{-1} = \infty .$$

$$\log|\zeta(\sigma+it)| = \sum_{p^m} \frac{1}{m} p^{-m\sigma} \cdot \mathrm{Re}(p^{-imt}), \quad \sigma > 1.$$

$$= \underline{\sum_p \frac{1}{p^\sigma} \cdot \mathrm{Re}(p^{-it})} + O(1)$$

$$\Downarrow$$

$$= \underline{-\sum_p \frac{1}{p^\sigma}} + \underbrace{\boxed{\overset{0}{\sum_p \frac{1 + \mathrm{Re}(p^{-it})}{p^\sigma}}}}_{O(1)} + O(1)$$

$p^{-it} \approx -1$

Mertens: $\sum_{p \le x} \frac{1}{p} = \log\log x + C + O\left(\frac{1}{\log x}\right).$

$$\sum_p \frac{1}{p^\sigma} = +\infty \quad \text{as } \sigma \to 1^+$$

$$\log|\zeta(\sigma+i\cdot 2t)| = \sum_p \frac{1}{p^\sigma}\cdot \mathrm{Re}(p^{-i2t}) + O(1)$$

$$= \underbrace{\sum_p \frac{1}{p^\sigma}}_{+\infty} - \underbrace{\boxed{\sum_p \frac{1-\mathrm{Re}(p^{-i2t})}{p^\sigma}}}_{O(1)} + O(1)$$

$$1-\mathrm{Re}(p^{-i2t}) \leq 2\left(1+\mathrm{Re}(p^{-it})\right)$$

$$\sum_p \frac{1-\mathrm{Re}(p^{-i2t})}{p^\sigma} \leq 2 \underbrace{\sum_p \frac{1+\mathrm{Re}(p^{-it})}{p^\sigma}}_{O(1)}$$

$$\mathbb{D}(I,f,g) := \left(\sum_{p\in I} \frac{1-\mathrm{Re}(f(p)\overline{g(p)})}{p}\right)^{\frac{1}{2}}, \quad |f(p)|, |g(p)| \leq 1.$$

$$f(p) = -1,\ g(p) = p^{it}$$

$$\mathbb{D}^{\mathbb{N}} := \{(x_n) \in \mathbb{C}^{\mathbb{N}} : |x_n| \leq 1\}.$$

$$(x_n)\cdot(y_n) := (x_n y_n)_{n=1}^{\infty}.$$

$\rho_n : D \to \mathbb{R}_{\geq 0}$ s.t. $\underline{\rho_n(zw) \leq \rho_n(z) + \rho_n(w)}$. $(*)$

$$\underline{\|(x_n)\|^2 = \sum_{n=1}^{\infty} \rho_n(x_n)^2 .}$$

"Δ-inequality": $\|(x_n)\times(y_n)\| \leq \|(x_n)\| + \|(y_n)\|$.

Check: $\rho_n(z)^2 = a_n \cdot (1 - \mathrm{Re}\, z)$ satisfies $(*)$.

$f : \mathbb{N} \to D$., $q_n = \{$ prime powers $\}$.
completely multiplicative

$\omega(n) = \#$ distinct prime powers of n.

$$\|(-1)^{\omega(q_n)} f(q_n)\|^2 = \sum_{n=1}^{\infty} a_n \cdot \left(1 - \mathrm{Re}\,(-1)^{\omega(q_n)}\overline{f(q_n)}\right)$$

$\|$

$$a_n = \underline{\frac{\Lambda(q_n)}{q_n^{\sigma} \log q_n}} = \frac{\log p}{p^{m\sigma}\log p^m} \quad \underline{\sum_{p^m} \frac{1}{m\, p^{m\sigma}}}\ \left(1 + \mathrm{Re}\, f(p^m)\right) = \underline{\log|\zeta(\sigma)| + \log|F(\sigma)|}$$

$q_n = p^m$

$$F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_p \left(1 - \frac{f(p)}{p^s}\right)^{-1}.$$

Prop 2. $\sqrt{\log|\zeta(\sigma)F(\sigma)|} + \sqrt{\log|\zeta(\sigma)G(\sigma)|} \geq \sqrt{\log \frac{\zeta(\sigma)}{H(\sigma)}}$, $\sigma > 1$

$$H(s) = \sum_{n=1}^{\infty} \frac{f(n)g(n)}{n^s} .$$

Take $f(n) = g(n) = n^{-it}$ s.t. $F(\sigma) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \zeta(\sigma+it)$, $H(\sigma) = \sum_{n=1}^{\infty} \frac{1}{n^{1+i2t}} = \zeta(\sigma+i2t)$.

Prop 2 $\Rightarrow$ $\zeta(\sigma)^3 |\zeta(\sigma+it)|^4 |\zeta(\sigma+i2t)| \geq 1$.

Halász Thm : $f: \mathbb{N} \to \mathbb{D}$. We have

$$\sum_{n \leq x} f(n) = o(x)$$

$\sum_{n\leq x} n^{it} \sim \frac{1}{1+it} \cdot x^{1+it} \neq o(x)$

$\Leftrightarrow$ $f(n)$ doesn't "pretend to be" n^{it} for any $t \in \mathbb{R}$:

$$\mathbb{D}([1,\infty), f, n^{it}) = \sum_{p} \frac{1-\mathrm{Re}(f(p)p^{-it})}{p} = \infty$$

for all $t \in \mathbb{R}$.

Pretentious Method for proving PNT :

- PNT $\Leftrightarrow$ $\sum_{n \leq x} \mu(n) = o(x)$, μ Möbius.

$$\Uparrow$$

- Prove that $\mu(n)$ doesn't pretend to be n^{it} for all $t \in \mathbb{R}$.

Sieve methods.